Observing the Dynamics of Supermassive Black Hole Binaries with Pulsar Timing Arrays

C. M. F. Mingarelli,[1] K. Grover,[1] T. Sidery,[1] R. J. E. Smith,[1,2] and A. Vecchio[1]
[1]School of Physics and Astronomy, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom
[2]Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
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Pulsar timing arrays are a prime tool to study unexplored astrophysical regimes with gravitational waves. Here, we show that the detection of gravitational radiation from individually resolvable supermassive black hole binary systems can yield direct information about the masses and spins of the black holes, provided that the gravitational-wave-induced timing fluctuations both at the pulsar and at Earth are detected. This in turn provides a map of the nonlinear dynamics of the gravitational field and a new avenue to tackle open problems in astrophysics connected to the formation and evolution of supermassive black holes. We discuss the potential, the challenges, and the limitations of these observations.

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Introduction.—Gravitational waves (GWs) provide a new means for studying black holes and addressing open questions in astrophysics and fundamental physics, from their formation, evolution, and demographics to the assembly history of galactic structures and the dynamical behavior of gravitational fields in the strong nonlinear regime. Specifically, GW observations through a network of radio pulsars used as ultrastable clocks—pulsar timing arrays (PTAs) [1–3]—represent the only *direct* observational avenue for the study of supermassive black hole binary (SMBHB) systems in the $\sim 10^8$–$10^9 M_\odot$ mass range, with orbital periods between ~ 1 month and a few years; see, e.g., [4,5] and references therein. Ongoing observations [6–9] and future instruments, e.g., the Square Kilometre Array [10], are expected to yield the necessary timing precision [11,12] to observe the diffuse GW background. This background is likely dominated by the incoherent superposition of radiation from the cosmic population of massive black holes [13–21], and within it we expect a handful of sources that are sufficiently close, massive, and high-frequency to be individually resolvable [22–30].

Massive black hole formation and evolution scenarios [31–34] predict the existence of a large number of SMBHBs. Furthermore, SMBHs are expected to be (possibly rapidly) spinning [35,36]. In fact, the dynamics of such systems—which, according to general relativity, are entirely determined by the masses and spins of the black holes [37]—leave a direct imprint on the emitted gravitational waveforms. From these, one could measure SMBHB masses and their distribution, yielding new insights into the assembly of galaxies and the dynamical processes in galactic nuclei [26]. Moreover, measuring the magnitude and/or orientation of spins in SMBHBs would provide new information on the role of accretion processes [38–42]. Finally, detection of SMBHBs could allow us to probe general relativistic effects in the nonlinear regime in an astrophysical context not directly accessible by other means; see [43] and references therein.

The observation of GWs with PTAs relies on the detection of the small deviation induced by gravitational radiation in the times of arrival (TOAs) of radio pulses from millisecond pulsars that function as ultrastable reference clocks. This deviation, called the residual, is the difference between the expected (without GW contribution) and actual TOAs once all the other physical effects are taken into account. The imprint of GWs on the timing residuals is the result of how the propagation of radio waves is affected by the GW-induced space-time perturbations along the travel path. It is a linear combination of the GW perturbation at the time when the radiation transits at a pulsar, the so-called "pulsar term," and then when it passes at the radio receiver, the "Earth term" [1–3]. The two terms reflect the state of a GW source at two *different* times of its evolution separated by $\tau \equiv (1 + \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{p}}) L_p \sim 3.3 \times 10^3 (1 + \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{p}})(L_p/1\ \mathrm{kpc})$ yr, where $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{p}}$ are the unit vectors that identify the GW propagation direction and the pulsar sky location at a distance L_p from Earth, respectively; see, e.g., [24]. (We use geometrical units in which $G = c = 1$.) In a network (array) of pulsars, all the perturbations at Earth add coherently and therefore boost the signal-to-noise ratio (S/N) of the signal. Each pulsar term is at a slightly different frequency since the orbital period of the binary evolves over the time τ.

Measuring the key physics of SMBHBs is hampered by the short (typically $T = 10$ yr) observation time compared to the typical orbital evolution time scale $f/\dot{f} = 1.6 \times 10^3 (\mathcal{M}/10^9 M_\odot)^{-5/3} (f/50\ \mathrm{nHz})^{-8/3}$ yr of binaries that are still in the weak-field adiabatic inspiral regime, with an orbital velocity $v = 0.12 (M/10^9 M_\odot)^{1/3} \times (f/50\ \mathrm{nHz})^{1/3}$ [44]. Here, $M = m_1 + m_2$, $\mu = m_1 m_2 / M$, and $\mathcal{M} = M^{2/5} \mu^{3/5}$ are the total, reduced, and chirp masses, respectively, of a binary with component masses $m_{1,2}$ and f is the GW emission frequency at the leading quadrupole order. The chirp mass determines the frequency evolution at the leading Newtonian order. In the post-Newtonian (pN) expansion of the binary evolution [45] in terms of $v \ll 1$, the second mass parameter enters at p^{1}N order [$\mathcal{O}(v^2)$ correction]; spins contribute at p$^{1.5}$N order and above [$\mathcal{O}(v^3)$], causing the orbital plane to precess through spin-orbit coupling, at leading order. These contributions are therefore seemingly out of observational reach.

The GW effect at the pulsar—the pulsar term—may be detectable in future surveys, and for selected pulsars their distance could be determined to subparsec precision [27,46,47]. If this is indeed the case, it opens the opportunity to coherently connect the signal observed at Earth and at pulsars, therefore providing snapshots of the binary evolution over $\sim 10^3$ yr. These observations would drastically change the ability to infer SMBHB dynamics and study the relevant astrophysical process and fundamental physics.

In this Letter, we show that, for SMBHBs at the high end of the mass and frequency spectrum observable by PTAs, say, $m_{1,2} = 10^9 M_\odot$ and $f = 10^{-7}$ Hz, the observations of a source still in the weak-field regime become sensitive to post-Newtonian contributions up to p$^{1.5}$N, including spin-orbit effects, if both the pulsar and Earth terms can be detected. This in principle enables the measurement of the two mass parameters and a combination of the spin's magnitude and relative orientation. We also show that the Earth term can be independently sensitive to spin-orbit coupling due to geometrical effects produced by precession. We discuss the key factors that enable these measurements and future observational prospects and limitations.

Signals from SMBHBs.—Consider a radio pulsar emitting radio pulses at frequency ν_0 in the source rest frame. GWs modify the rate at which the radio signals are received at Earth [1–3], inducing a relative frequency shift $\delta\nu(t)/\nu_0 = h(t-\tau) - h(t)$, where $h(t)$ is the GW strain. The quantities that are actually produced at the end of the data reduction process of a PTA are the timing residuals, $\int dt' \delta\nu(t')/\nu_0$, although, without loss of generality, we will base the discussion on $h(t)$. The perturbation induced by GWs is repeated twice and carries information about the source at time t, the "Earth term," and past time $t - \tau$, the "pulsar term."

We model the radiation from a SMBHB using the so-called restricted pN approximation, in which pN corrections are included only in the phase and the amplitude is retained at the leading Newtonian order, but we include the leading order modulation effects produced by spin-orbit coupling. The strain is given by

$$h(t) = -A_{\rm gw}(t) A_p(t) \cos[\Phi(t) + \varphi_p(t) + \varphi_T(t)], \qquad (1)$$

where $A_{\rm gw}(t) = 2[\pi f(t)]^{2/3} \mathcal{M}^{5/3}/D$ is the Newtonian order GW amplitude; $\Phi(t)$ is the GW phase—see, e.g., Eqs. (232) and (234) in [45] and Eq. (8.4) in [48]—and D is the distance to the GW source. $A_p(t)$ and $\varphi_p(t)$ are the time-dependent polarization amplitude and phase and $\varphi_T(t)$ is an additional phase term, analogous to Thomas precession; see Eq. (29) in [49].

The physical parameters leave different observational signatures in the GW strain $h(t)$ and are therefore found in the TOA residuals. At the leading Newtonian order, $\mathcal{M}$ drives the frequency and therefore the phase $\Phi(t)$ evolution, with the second independent mass parameter entering from the p^{1}N onwards. SMBHs are believed to be rapidly spinning, and the spins are responsible for three distinctive imprints in the waveform: (i) they alter the phase evolution through spin-orbit coupling and spin-spin coupling at p$^{1.5}$N and p^{2}N order, respectively [50]; (ii) they cause the orbital plane to precess due to (at lowest order) spin-orbit coupling and therefore induce amplitude and phase modulations in the waveform through $A_p(t)$ and $\varphi_p(t)$; and (iii) through orbital precession, they introduce an additional secular contribution $\varphi_T(t)$ to the waveform phase. Astrophysically, we expect PTAs to detect SMBHBs of comparable component masses [24]. We therefore model the spin-orbit precession using the *simple precession approximation* [49], which formally applies when $m_1 = m_2$ or when one of the two spins is negligible with respect to the other. Let $\mathbf{S}_{1,2}$ and $\mathbf{L}$ be the black holes' spins and the orbital angular momentum, respectively. Then, both $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ and $\mathbf{L}$ precess around the (essentially) constant direction of the total angular momentum, $\mathbf{J} = \mathbf{S} + \mathbf{L}$, at the same rate $d\alpha/dt = \pi^2[2 + 3m_2/(2m_1)](|\mathbf{L} + \mathbf{S}|) f^2(t)/M$ [49], where α is the precession angle, while preserving the angle of the precession cone, λ_L; see Fig. 4 of Ref. [49]. This approximation is adequate to conceptually explore these effects; however, in the case of real observations, one will need to consider the exact expressions [51].

The detection and particularly the measurement of the aforementioned parameters rely on coherently matching the signal with a template that faithfully reproduces its amplitude and, importantly, its phase evolution. We therefore consider the contribution to the total number of wave cycles a proxy for the significance of a specific parameter. Individual terms that contribute ~ 1 GW cycle or more mean that the effect is in principle detectable; hence, one can infer information about the associated parameter(s). We show that information about the parameters can only be inferred for SMBHBs at the high end of the mass spectrum and PTA observational frequency range. Having a sufficiently high-mass and high-frequency GW source is also essential to ensuring sufficient frequency evolution over the time τ, so that the Earth and pulsar terms are clearly separated in frequency space; cf. Table I. We therefore take fiducial source parameters of $m_1 = m_2 = 10^9 M_\odot$, frequency at Earth at the beginning of the observation $f_E = 10^{-7}$ Hz, and an observational time of $T = 10$ years to illustrate the main results. We provide scaling relations as a function of the relevant quantities, allowing the reader to rescale the results for different astrophysical and/or observational values.

TABLE I. Frequency change Δf, total number of GW cycles, and individual contributions from the leading order terms in the pN expansion over the two relevant time scales—a 10 yr period starting at Earth and the time period $L_p(1 + \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{p}})$ between the Earth and pulsar terms (hence the negative sign)—for selected values of $m_{1,2}$ and f_E.

m_1 ($M_\odot$)	m_2 ($M_\odot$)	f_E (nHz)	($v/c \times 10^{-2}$)	Time span	Δf (nHz)	Total	Newtonian	p^{1}N	p$^{1.5}$N	Spin orbit/β	p^{2}N
10^9	10^9	100	14.6	10 yr	3.22	32.1	31.7	0.9	−0.7	0.06	0.04
			9.6	−1 kpc	71.2	4305.1	4267.8	77.3	−45.8	3.6	2.2
		50	11.6	10 yr	0.24	15.8	15.7	0.3	−0.2	0.01	<0.01
			9.4	−1 kpc	23.1	3533.1	3504.8	53.5	−28.7	2.3	1.2
10^8	10^8	100	6.8	10 yr	0.07	31.6	31.4	0.2	−0.07	<0.01	<0.01
			6.4	−1 kpc	15.8	9396.3	9355.7	58.3	−19.9	1.6	0.5
		50	5.4	10 yr	0.005	15.8	15.7	0.06	−0.02	<0.01	<0.01
			5.3	−1 kpc	1.62	5061.4	5045.8	20.8	−5.8	0.5	0.1

Observations using the Earth term only.—We start by considering analyses that rely only on the Earth term contribution to the residuals, as done in Refs. [25,52]. The case of a coherent analysis based both on the Earth and pulsar terms, introduced in Ref. [22], is discussed later in this Letter. Table I shows that, in general, the frequency change over 10 yrs is small compared to the frequency bin width, 3.2(10 yr$/T$) nHz [24,27]. The observed signal is effectively monochromatic, making the dynamics of the system impossible to infer. However, the presence of spins affects the waveform not only through the GW phase evolution, but also via the modulations of $A_p(t)$ and $\varphi_p(t)$ that are periodic over the precession period, *and* also introduces the secular contribution $\varphi_T(t)$. For $m_{1,2} = 10^9 M_\odot$ and $f_E = 10^{-7}$ Hz, the orbital angular momentum precesses by $\Delta\alpha = 2$ rad (for dimensionless spin parameter $a \equiv S/M^2 = 0.1$) and $\Delta\alpha = 3$ rad (for $a = 0.98$), and therefore the additional modulation effect on $A_p(t)$ and $\varphi_p(t)$ is small and likely undetectable. However, the overall change of $\varphi_T(t)$ over 10 yrs could be appreciable: the average contribution for each precession cycle of this additional phase term is $\langle\Delta\varphi_T\rangle = 4\pi$ or $4\pi(1 - \cos\lambda_L)$, depending on whether $\hat{\mathbf{\Omega}}$ lies inside or outside the precession cone, respectively [49]. If $\hat{\mathbf{\Omega}}$ lies inside the precession cone, and given that the observation will cover between a third and a half of a full precession cycle, then $\langle\Delta\varphi_T\rangle \sim \pi$, which could surely indicate the presence of spins. On the other hand, the precession cone will be small in general since $|S/L| \sim av(M/\mu) \simeq 0.1a(M/\mu)(M/10^9 M_\odot)^{1/3}(f/100\text{ nHz})^{1/3}$; therefore, the likelihood of $\hat{\mathbf{\Omega}}$ lying inside the precession cone is small, assuming an isotropic distribution and orientation of sources. In this case, the Thomas precession contribution (per precession cycle) is suppressed by a factor $(1-\cos\lambda_L) \simeq \lambda_L^2/2 \sim 5\times10^{-3}a^2(M/\mu)^2(M/10^9 M_\odot)^{2/3}\times(f/100\,\text{nHz})^{2/3}$, which will produce a negligible contribution $\Delta\varphi_T(t) \ll 1$. However unlikely, spins may still introduce observable effects that need to be taken into account.

Measuring SMBHB evolution using the Earth and pulsar terms.—With more sensitive observations and the increasing possibility of precisely determining L_p—see, e.g., [47]—the prospect of also observing the contribution from the pulsar term from one or more pulsars becomes more realistic. We show below that, *if* at least one of the pulsar terms can be observed together with the Earth term, this opens opportunities to study the dynamical evolution of SMBHBs and, in principle, to measure their masses and spins. This is a straightforward consequence of the fact that PTAs become sensitive to $\sim10^3$ yrs of SMBHB evolutionary history, in "snippets" of length $T \ll L_p$ that can be coherently concatenated.

The signal from each pulsar term will be at a S/N which is significantly smaller than the Earth term by a factor $\sim\sqrt{N_p}$, where N_p is the number of pulsars that effectively contribute to the S/N of the array. For example, if the Earth term yields a S/N of $\sim36\sqrt{N_p/20}$, then each individual pulsar term would give a $S/N \sim 8$. The possibility of coherently connecting the Earth term signal with each pulsar term becomes therefore a question of S/N, prior information about the pulsar-Earth baseline, and how accurately the SMBHB location in the sky can be reconstructed, as part of a "global fit"; see, e.g., [27]. Assuming for simplicity that the uncertainties on L_p and $\hat{\mathbf{\Omega}}$ are uncorrelated, this requires that the distance to the pulsar and the location of the GW source are known with the errors $\lesssim 0.01(100\text{ nHz}/f)$ pc and $\lesssim 3(100\text{ nHz}/f)\times(1\text{ kpc}/L_p)$ arcsec, respectively. These are very stringent constraints [24,28,47], and a detailed analysis is needed in order to assess the feasibility of reaching this precision. Clearly, if an electromagnetic counterpart to the GW source were to be found [53,54], it would enable the identification of the source location in the sky, making the latter constraint unnecessary. We can now consider the contribution from the different terms in the pN expansion to the total number of cycles in observations that cover the GW source evolution over the time τ that are encoded in the simultaneous analysis of the Earth *and* pulsar terms. The results are summarized in Table I, for selected values of $m_{1,2}$ and f_E and for a fiducial value $\tau = 1$ kpc. The wave cycle contributions from the spin-orbit parameter are normalized to $\beta = (1/12)\sum_{i=1}^{2}[113(m_i/M)^2 + 75\eta]\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_i$, which has a maximum value of 7.8. Contributions from the p^{2}N order spin-spin terms are negligible. The results clearly show that, despite the fact that the source is in the weak-field regime, the extended Earth-pulsar baseline requires the p$^{1.5}$N and in some rare cases the p^{2}N contribution to accurately (i.e., within $\sim$1 GW cycle) reproduce the full phase evolution.

For $m_{1,2} = 10^9 M_\odot$ and $f_E = 10^{-7}$ Hz, there is a total of 4305 GW cycles over a 1 kpc light travel time evolution, with the majority (4267) accounted for by the leading order Newtonian term, providing information about the chirp mass, and tens of cycles due to the p^{1}N and p$^{1.5}$N terms (77 and 45, respectively), which provide information about a second independent mass parameter. Spins contribute to phasing at p$^{1.5}$N with $\sim 3\beta$ cycles. Therefore, their total contribution is smaller than the p$^{1.5}$N mass contribution by a factor between a few and $\sim$10. The additional Thomas precession phase contribution may become comparable to the p^{1}N mass contribution in some cases. In fact, for $a = 0.1(0.98)$, the binary undergoes 24 (34) precession cycles. This corresponds to a total Thomas precession phase contribution of 306 (426) rad if $\hat{\mathbf{\Omega}}$ lies outside the precession cone.

The modulations of $A_p(t)$ and $\varphi_p(t)$ are characterized by a small λ_L, because for most of the inspiral $S \ll L$, and are likely to leave a smaller imprint on the waveform than those discussed so far. We can indeed estimate the importance of this effect for the most favorable parameter combinations. The value of $\varphi_p(t)$ oscillates over time with an amplitude which depends on the time to coalescence, $\mathbf{S}$, $\mathbf{L}$, $\hat{\mathbf{\Omega}}$, and $\hat{\mathbf{p}}$. We choose the orientation of $\hat{\mathbf{S}}$ such that λ_L is maximized, and we vary $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{p}}$, each of which is drawn from a uniform distribution on the two-sphere.

In Fig. 1, we show that, for rapidly spinning ($a = 0.98$) SMBHBs, this effect could introduce modulations larger than $\pi/2$ in $\varphi_p(t)$ over 30% of the parameter space of possible $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{p}}$ geometries. The amplitude would correspondingly change over the same portion of the parameter space by at most 60% with respect to its unmodulated value. Since these effects are modulated, they will not be easily identifiable.

Conclusions.—We have established that the coherent observation of both the Earth and pulsar terms provides information about the dynamical evolution of a GW source. The question now is whether they can be unambiguously identified. A rigorous analysis would require extensive simulations based on the actual analysis of synthetic data sets. We can, however, gain the key information with a much simpler order of magnitude calculation. The phase (or number of cycles) error scales as $\sim 1/(S/N)$. Assuming $S/N \sim 40$ means that the total number of wave cycles over the Earth-pulsar baseline can be determined with an error of $\sim 4300/40 \sim 100$ wave cycles.

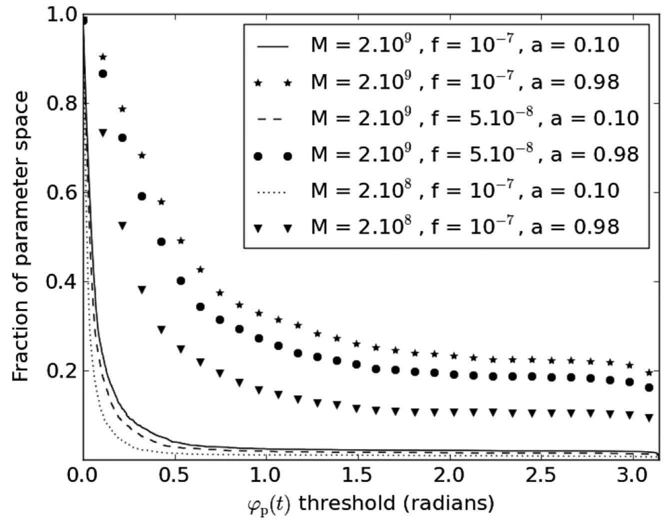

FIG. 1. The fraction of parameter space in $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{p}}$ for which the maximum excursion of φ_p over the time $L_p(1 + \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{p}})$ for $L_p = 1$ kpc exceeds a certain value, shown on the horizontal axis. Several values of $m_{1,2}$, a, and f_E are considered (see the legend).

This is comparable to the p^{1}N contribution to the GW phase and, in very favorable circumstances, to the Thomas precession phase contribution and larger by a factor of a few or more than all the other contributions. It may therefore be possible to measure the chirp mass and, say, the symmetric mass ratio of a SMBHB and possibly a combination of the spin parameters. Effects due to the p$^{1.5}$N and higher phase terms are likely to remain unobservable, as well as amplitude and phase modulations. Correlations between the parameters, in particular, masses and spins, will further degrade the measurements. The details will depend on the actual S/N of the observations, the GW source parameters, and the accuracy with which the source location and the pulsar distance can be determined. We plan to explore these issues in detail in a future study.

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