Schwarzschild Black Holes can Wear Scalar Wigs

Juan Barranco,[1] Argelia Bernal,[2] Juan Carlos Degollado,[3] Alberto Diez-Tejedor,[1] Miguel Megevand,[2] Miguel Alcubierre,[2]
Darío Núñez,[2] and Olivier Sarbach[4]

[1]*Departamento de Física, División de Ciencias e Ingeniería, Campus León, Universidad de Guanajuato, León 37150, Mexico*
[2]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior Ciudad Universitaria,
A.P. 70-543, México Distrito Federal 04510, Mexico*
[3]*Instituto de Astronomía, Universidad Nacional Autónoma de México, Circuito Exterior Ciudad Universitaria,
A.P. 70-264, México Distrito Federal 04510, Mexico*
[4]*Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3,
Ciudad Universitaria, 58040 Morelia, Michoacán, Mexico*
(Received 11 May 2012; published 22 August 2012)

We study the evolution of a massive scalar field surrounding a Schwarzschild black hole and find
configurations that can survive for arbitrarily long times, provided the black hole or the scalar field mass is
small enough. In particular, both ultralight scalar field dark matter around supermassive black holes and
axionlike scalar fields around primordial black holes can survive for cosmological times. Moreover, these
results are quite generic in the sense that fairly arbitrary initial data evolve, at late times, as a combination
of those long-lived configurations.

Introduction.—It has been known for some time that a
Schwarzschild black hole (BH) cannot support a nontrivial
scalar field (SF) distribution; i.e., such a hole does not
admit scalar hair [1,2]. However, the no-hair theorems do
not exclude the existence of dynamical solutions that decay
very slowly in time. In a recent work [3] we found regular
SF configurations surrounding a Schwarzschild BH that
can survive for relatively long times, and conjectured that,
for a certain range of values of the SF and BH masses, such
SF configurations could survive in the vicinity of the BH
for cosmological time scales. Here we show that this is
indeed what happens, and that such distributions arise from
the evolution of fairly arbitrary initial configurations. Our
results give support to the idea that the dark matter halos
could be described by a coherent scalar excitation [4,5]
(see Refs. [6,7] for papers with a similar motivation), and
to the possibility of nontrivial axion distributions surround-
ing primordial BHs [8].

Three types of resonances.—The dynamics of a SF
propagating on a Schwarzschild background are described
by the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial r_*^2} + V(r)\phi = 0, \tag{1a}$$

with $V(r)$ an effective potential given by

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right). \tag{1b}$$

Here M is the BH mass, ℓ the angular momentum number
(we decompose the field in spherical harmonics), μ the
mass of the SF (in units for which $G = c = \hbar = 1$), and r_*
$(-\infty < r_* < +\infty)$ the radial "tortoise" coordinate, re-
lated to the Schwarzschild radial coordinate r $(r > 2M)$

by $r_* = r + 2M \ln(r/2M - 1)$. As discussed in detail in
Ref. [3] (see also Ref. [9]), for each $\ell \geq 0$ and $M\mu <
M\mu_{\text{crit}}(\ell)$, V describes a potential well which is enclosed
between a barrier close to the BH, and the asymptotic
positive value μ^2 of the potential at infinity.

The stationary solutions of Eq. (1) are characterized by a
time dependency of the form $e^{i\omega t}$ with a real frequency ω.
Such solutions behave as a combination of ingoing and
outgoing waves to the left of the potential barrier, close to
the horizon, and we require them to decay exponentially to
zero at spatial infinity. There is a continuous spectrum of
such solutions for $\omega^2 < \mu^2$. Furthermore, as we have
shown in Ref. [3], for ω lying in the region $V_{\text{min}} < \omega^2 <
\mu^2$, with V_{min} the local minimum of the potential, there is a
discrete set of such frequencies for which the amplitude
inside the potential well takes very large values when
compared with the amplitude close to the horizon. We
call these modes the *stationary resonances*, and the region
$V_{\text{min}} < \omega^2 < \mu^2$ the *resonance band*.

If one imposes the condition of no waves coming from
the region close to the horizon, while keeping the require-
ment of exponential decay at spatial infinity, it turns out
that Eq. (1) is only satisfied for a discrete set of complex
frequencies. These solutions have been called *quasireso-
nances* in the literature [10] (see also Ref. [11] for a
previous study of such solutions).

Both the stationary resonances and quasiresonances are
in fact nonphysical solutions. First, as mentioned above, all
purely stationary solutions, i.e., those with real ω, require
waves to move outward from the horizon region to com-
pensate for the waves that tunnel out through the barrier
and move toward the horizon (as otherwise the situation
would not be stationary). Imposing the condition of no

waves coming out from the horizon clearly improves the situation at the cost of introducing complex valued frequencies ω. This makes sense physically since now the solutions must decay in time as the waves tunnel out of the potential well and fall toward the BH, with this decay represented by the imaginary part of the frequency. Nevertheless, it turns out that such solutions are still nonphysical, as one can show that the energy density diverges at the horizon for both types of solutions.

There is yet a third class of resonant solutions we will consider. In Ref. [3] we performed numerical evolutions of regular initial data that have not such a divergence of energy density at the horizon, and found damped oscillating solutions that remain surrounding the BH for long times. These solutions have a frequency equal (within the numerical error) to that of the stationary resonances. We call these solutions *dynamical resonances*. We will show that the real part of the frequency of the quasiresonant modes coincides with the frequency of oscillation of the stationary and dynamical resonances, and that the imaginary part coincides with the decay rate of the dynamical resonances. This is a nontrivial result since, after all, the dynamical solutions are in some sense "infinitely different" to both the stationary resonances and the quasiresonances because they are regular solutions with finite energy. This is similar to the case of the quasinormal modes of BHs, which also diverge at the horizon, but nevertheless physical excitations behave as combinations of them locally at late times.

Toy model.—In order to illustrate how the different resonant modes arise and are related to each other, we first consider a simple toy model in which the potential V in Eq. (1b) is replaced by

$$U(x) = A\delta(x) + \mu^2\Theta(x - a), \qquad x := r_*, \quad (2)$$

with A and a positive constants. That is, we have a potential well enclosed between the δ barrier and a step function where U jumps from zero to its asymptotic value μ^2. For $M\mu \ll 1$ and $\ell \geq 1$ the minimum of the potential well of the physical potential V is located at $r_{\min} = \ell(\ell + 1) \times (M\mu^2)^{-1}[1 + \mathcal{O}(M\mu)^2]$, and the strength of its barrier is $\int_{-\infty}^{r_{*\min}} V dr_* = [6\ell(\ell + 1) + 1](4M)^{-1}[1 + \mathcal{O}(M\mu)^2]$. Therefore, we require the dimensionless parameters $\lambda := 2\mu/A$ and $p := (a\mu)^{-1}$ to scale like $M\mu$ in our toy model potential U. In the analysis below we prove that for fixed $p < 1/\pi$ and small enough values of λ, the potential U gives rise to a discrete set of stationary and quasiresonant frequencies whose real parts agree with each other.

The resonant modes for the potential U have the form

$$\phi(t, x) = e^{i\omega t} \times \begin{cases} \alpha e^{i\omega x} + \beta e^{-i\omega x}, & x < 0, \\ \gamma e^{i\omega x} + \delta e^{-i\omega x}, & 0 < x < a, \\ e^{-\sqrt{\mu^2 - \omega^2}x}, & x > a, \end{cases} \quad (3)$$

with ω a complex frequency. We require $\omega \notin (-\infty, -\mu] \cup [\mu, \infty)$, which guarantees that $\mathrm{Re}\sqrt{\mu^2 - \omega^2} \neq 0$, and

therefore we can choose the sign of the square root such that its real part is positive, leading to exponential decay for $x > a$, as required. The constants α, β, γ, and δ are determined by the matching conditions at $x = 0$ and $x = a$ which consist in the continuity of ϕ and its first spatial derivative ϕ' at $x = a$ and the continuity of ϕ and the jump condition $\phi'(t, 0^+) - \phi'(t, 0^-) = A\phi(t, 0)$ at $x = 0$.

For the stationary resonances ω is real and our restrictions imply $\omega \in (-\mu, \mu)$. Since U is real, it follows in this case that $\alpha = \beta^*$, $\gamma = \delta^*$, and the ratio between the amplitudes of the solution for $x < 0$ and the one inside the well is $\nu(\omega) = |\alpha|/|\gamma|$. The resonance frequencies are the ones that minimize this ratio. For the quasiresonances, in turn, the condition of no waves coming from the horizon implies that $\beta = 0$, which can only occur for a nontrivial imaginary part of ω. In terms of the quantities λ and p defined below Eq. (2), and the complex angle φ lying inside the strip $-\pi/2 < \mathrm{Re}(\varphi) < \pi/2$ defined by $\omega = \mu \sin\varphi$, the condition $\beta = 0$ reads

$$F_p(\lambda, \varphi) := 1 + i\lambda \sin\varphi - e^{-2i\varphi - (2i/p)\sin\varphi} = 0, \quad (4)$$

while the ratio of interior and exterior solutions in the stationary case is

$$\nu(\omega) = \frac{1}{\lambda} \frac{|F_p(\lambda, \varphi)|}{\sin\varphi}. \quad (5)$$

As we prove now there exists, for each $p < 1/\pi$ and small enough λ, a finite family $\varphi_n(\lambda)$ of solutions of Eq. (4) with the following properties: they depend smoothly on λ, their imaginary parts decay like λ^2, and their real parts agree with the local minima of $\nu(\omega)$. This shows the existence of quasiresonant modes for the toy model potential U whose frequencies are related to those of the stationary resonant modes through their real parts.

In order to show this we consider first the limit $\lambda = 0$, in which case Eq. (4) yields real solutions $\varphi_n^{(0)} \in (-\pi/2, \pi/2)$ satisfying $\sin\varphi_n^{(0)} = p(n\pi - \varphi_n^{(0)})$, with $n = 1, 2, 3, \ldots$. There are a finite number of solutions when $p < 1/\pi$. For $p \ll 1/\pi$ the fundamental solution is approximately $\varphi_1^{(0)} \approx p\pi$. In order to extend these solutions for small $\lambda > 0$ we invoke the implicit function theorem and observe that the function $F_p(\lambda, \varphi)$ is linear in λ and analytic in φ. Furthermore, $\frac{\partial F_p}{\partial \varphi}(0, \varphi_n^{(0)}) = 2i(1 + p^{-1}\cos\varphi_n^{(0)}) \neq 0$ since $-\pi/2 < \varphi_n^{(0)} < \pi/2$. Therefore, there exists for small enough λ a unique smooth solution curve $\varphi_n(\lambda)$ satisfying $F_p[\lambda, \varphi_n(\lambda)] = 0$ and $\varphi_n(0) = \varphi_n^{(0)}$. A Taylor expansion reveals that the corresponding complex frequencies $\omega_n(\lambda) = \mu \sin[\varphi_n(\lambda)]$ have the form

$$\mathrm{Re}[\omega_n(\lambda)] = \mu \sin\varphi_n^{(0)}\left[1 - \frac{\lambda}{q_n} + \frac{\sigma}{q_n^2}\lambda^2 + \mathcal{O}(\lambda^3)\right], \quad (6a)$$

$$\mathrm{Im}[\omega_n(\lambda)] = \frac{\mu}{2}\frac{\sin^2\varphi_n^{(0)}}{q_n}\lambda^2 + \mathcal{O}(\lambda^3), \quad (6b)$$

where we have defined $q_n := 2(\cos^{-1}\varphi_n^{(0)} + p^{-1})$ and $\sigma := 1 - q_n^{-1}\sin^2\varphi_n^{(0)}/\cos^3\varphi_n^{(0)}$. Furthermore, one can verify that the local minima of the function $\nu(\omega)$ given in Eq. (5), describing the stationary resonances, have precisely the expansion given in Eq. (6a). Therefore, up to second order in λ, the frequencies of the stationary resonances agree with the real part of the quasiresonant frequencies.

Since both parameters λ and p scale like $M\mu$ in our model potential, it follows that for $M\mu \ll 1$ the imaginary part of $M\omega_n$ scales like $(M\mu)^6$. This explains, at least qualitatively, why the quasiresonant modes are associated with such large time scales when $M\mu \ll 1$.

It is also possible to relate the quasiresonant frequencies ω_n to the dynamical resonances by representing the solutions of Eq. (1) as an inverse Laplace integral. This calculation will be presented elsewhere.

Frequency comparison and characteristic times.—For the physical potential, Eq. (1b), the quasiresonant modes can be obtained semianalytically in several ways, the most common are the continued fraction method introduced by Leaver [12], and the WKB approach [13] (see Refs. [14–17] for details). An analytic expression valid for small values of the combination $M\mu$ was obtained by Detweiler in Ref. [11]. The stationary and dynamical modes were previously considered in Ref. [3]. In this section we show that the oscillations of the dynamical resonances have frequencies equal (up to numerical error) to the real part of the frequencies of the quasiresonant modes, while the decay rates are equal to the imaginary part of such frequencies (see Fig. 1). We also show that the imaginary part of the quasiresonant frequencies calculated by means of Leaver's method has the limit behavior predicted by Detweiler. Hence, we can use the Detweiler expressions to determine the decay rate of the dynamical resonances with very small values of $M\mu$, which cannot be reached with the other two methods.

Leaver's semianalytical approach [12] assumes a harmonic time dependence for the SF, $\phi(t, r) = \psi(r)e^{i\omega t}$, with ω the complex quasiresonant frequency, and

FIG. 1 (color online).   The imaginary part of the frequency for the first quasiresonant mode obtained using Leaver's method is shown as a function of $M\mu$ for the cases $\ell = 0, 1$.

proposes a power series expansion for the radial function $\psi(r)$ of the form

$$\psi = \tilde{\psi} \sum_{n=0}^{\infty} a_n \left(1 - \frac{2M}{r}\right)^n, \qquad (7)$$

where, following Ref. [18], we define $\tilde{\psi} := e^{-\chi r} r^{(-2M\chi + M\mu^2/\chi)}(1 - 2M/r)^{-2iM\omega}$. This guarantees that close to the horizon $\psi \sim e^{i\omega r^*}$, whereas at spatial infinity $\psi \sim e^{-\chi r^*} r^{M\mu^2/\chi}$, with $\chi = \sqrt{\mu^2 - \omega^2}$ and the sign convention for the square root as explained in the previous section. Notice however that our boundary conditions are different than those in Ref. [18].

In Fig. 1 we plot the imaginary part of the frequency of the first quasiresonant mode (the one with the lower value of $\text{Re}(M\omega)$) for the cases $\ell = 0, 1$ and different values of $M\mu$. We find that the real part of the frequencies obtained corresponds to the frequencies at which the SF oscillates in a dynamical scenario [3]. Furthermore, we also find that the imaginary part of the frequencies (dots in Fig. 1) corresponds to the decay rate of the field in the numerical evolutions (empty circles in Fig. 1).

In the numerical evolutions we were not able to evolve the configurations for small values of $M\mu$, both because of the propagation of the errors in the numerical approximation, and because the time required for such evolutions becomes prohibitive. Even though Leaver's method allows us to obtain results for parameter values that could not be reached with the numerical evolutions, numerical roundoff errors make it still prohibitive to obtain accurately very small values for the imaginary part of the quasiresonant frequencies.

In Ref. [11] analytic expressions for the spectrum of quasiresonant modes were found in the limit $M\mu \ll 1$, for values of $\ell \geq 1$. For $\ell = 1$, the imaginary part of the frequency for the first quasiresonant mode in this limit is given by $\text{Im}(M\omega) = (M\mu)^{10}/6$. Strictly speaking, the expressions given in Ref. [11] may not apply for the case with $\ell = 0$. However, if we nevertheless use them in that case we find $\text{Im}(M\omega) = 16(M\mu)^6$. We have calculated the quasiresonant mode frequencies using Leaver's method up to the smallest values of $M\mu$ allowed by the roundoff errors, and we have found that the imaginary part of such frequencies matches the Detweiler approximations (within errors) at small values of $M\mu$ (see Fig. 1). This result allows us to conclude that, for small values of $M\mu$, the decay time of the dynamical resonances should correspond to the one predicted by Detweiler in Ref. [11].

Since the decay rate of the dynamical resonances is related to the imaginary part of the quasiresonant frequencies, then its half-life time is inversely proportional to $\text{Im}(M\omega)$. In particular we can find BH and SF masses such that $t_{1/2} \sim 10^{10}$ years and larger for the cases $\ell = 0$, 1 previously mentioned. Those results are shown in Fig. 2. There are two distinct regions of the parameter space of
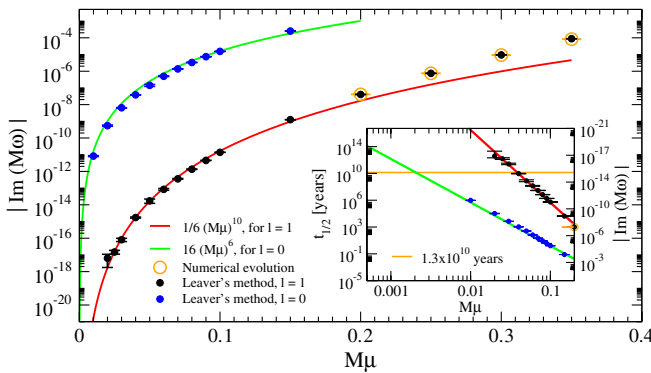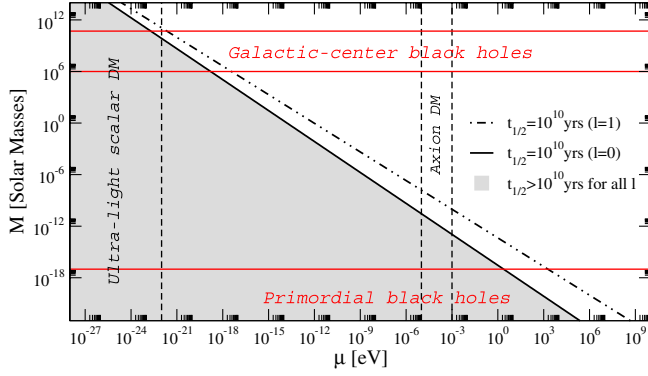
FIG. 2 (color online).   Combinations of the parameters M and μ for which the SF half-life time can be larger than the age of the Universe.
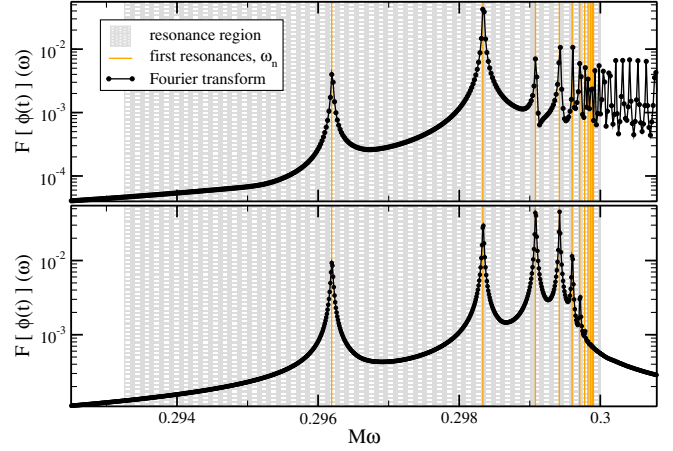

FIG. 3 (color online).   Fourier transform in time of the evolution of initial data with $R_1 = 100$, $R_2 = 200$ (top panel); and $R_1 = -1000$, $R_2 = 1000$ (bottom panel). The peaks to the right of the resonance band in the top panel are a numerical artifact caused by noise originating at the outer boundary.

physical interest for which the configurations live longer than the age of the Universe: (i) a SF mass smaller than 1 eV and BH mass smaller than $10^{-17}M_\odot$, consistent with primordial BHs with an axion distribution [8]; and (ii) an ultralight (fuzzy) SF [19–21] with a mass smaller than 10^{-22} eV and a supermassive BH with a mass smaller than $5 \times 10^{10}M_\odot$, as could be the case for a dark matter halo [4,5] surrounding a BH at a Galactic center.

Evolution of arbitrary configurations.—In this section we study the long-term numerical evolution of SF distributions that are initially surrounding a Schwarzschild BH. We show that at late times, after some of the SF falls into the BH and some escapes to infinity, the remaining SF consists of a superposition of the dynamical resonances, which, for practical purposes, can be described in terms of the quasiresonant modes. Remarkably, we see that this holds even for pretty arbitrary initial data. Note that the calculations in this section are identical to those in Ref. [3], but with the crucial difference that here they are performed on arbitrary initial data (as opposed to initial data obtained from stationary resonances), hence leading to new, much stronger results.

In order to test the evolution of a variety of SF distributions, we construct a two-parameter family of initial data, of the form

$$u_0(r) = \begin{cases} N(r - R_1)^4(r - R_2)^4 & \text{for } R_1 \leq r \leq R_2 \\ 0 & \text{otherwise} \end{cases}, \quad (8)$$

with the normalization $N = [2/(R_1 - R_2)]^8$. The parameters R_1 and R_2 are chosen in order to set distributions at $t = 0$ with different "locations" and "sizes." The initial value of the time derivative, $\dot{u}|_{t=0}$, is constructed as

$$\dot{u}|_{t=0} = \frac{\delta f}{\delta t}\bigg|_{t=0}, \qquad f(t, r) := u_0(r - vt), \quad (9)$$

where v is a free parameter that has no significant effect on the results presented here, and for the majority of configurations studied is simply set to zero. The evolution equation is solved numerically, as described in Ref. [22].

As expected, during the initial stages of the evolution some of the SF accretes into the BH, while some is radiated away, both with rates that depend greatly on the initial data chosen, and can be very large in some cases. However, at late times, all evolutions show a similar steady behavior with slow accretion into the BH. Similar results are obtained when studying the long-term evolution of a variety of configurations, some of them very different in size and spatial distribution—although all distributions studied here are quite "wide" when compared to the BH size.

One can gain some understanding of this behavior by conducting a spectral analysis as follows. We perform a (discrete) Fourier transform in time, $\mathcal{F}[\phi(t, r_j)](\omega)$, of the SF at several (about 20) sample points r_j, then take the average to obtain the spectrum $\mathcal{F}[\phi(t)](\omega)$. For brevity, we present here results corresponding to only two representative examples. They correspond to initial data with $R_1 = 100M$, $R_2 = 200M$, and $R_1 = -1000M$, $R_2 = 1000M$. In both cases $\mu M = 0.3$, $\ell = 1$, and $v = 0$. The spectra are shown in Fig. 3.

For comparison, we have indicated in the figure the first resonance frequencies (vertical lines), as obtained in Ref. [3]. One can clearly see peaks that coincide with each resonant frequency. These results seem to indicate that, at late times, even quite generic SF distributions evolve as a combination of the resonant modes, which, as we have shown, can last for cosmological time scales.

[1] J. D. Bekenstein, Phys. Rev. D **51,** R6608 (1995).
[2] I. Pena and D. Sudarsky, Classical Quantum Gravity **14,** 3131 (1997).
[3] J. Barranco, A. Bernal, J. C. Degollado, A. Diez-Tejedor, M. Megevand, M. Alcubierre, D. Núñez, and O. Sarbach, Phys. Rev. D **84,** 083008 (2011).
[4] S.-J. Sin, Phys. Rev. D **50,** 3650 (1994).
[5] A. Arbey, J. Lesgourgues, and P. Salati, Phys. Rev. D **64,** 123528 (2001).
[6] L. A. Urena-Lopez and A. R. Liddle, Phys. Rev. D **66,** 083005 (2002).
[7] A. Cruz-Osorio, F. S. Guzman, and F. D. Lora-Clavijo, J. Cosmol. Astropart. Phys. 06 (2011) 029.
[8] P. Sikivie and Q. Yang, Phys. Rev. Lett. **103,** 111301 (2009).
[9] J. Grain and A. Barrau, Eur. Phys. J. C **53,** 641 (2008).
[10] A. Ohashi and M.-a. Sakagami, Classical Quantum Gravity **21,** 3973 (2004).
[11] S. Detweiler, Phys. Rev. D **22,** 2323 (1980).
[12] E. Leaver, Proc. R. Soc. A **402,** 285 (1985).
[13] S. Iyer and C. M. Will, Phys. Rev. D **35,** 3621 (1987).
[14] H.-P. Nollert, Classical Quantum Gravity **16,** R159 (1999).
[15] K. D. Kokkotas and B. Schmidt, Living Rev. Relativity **2,** 2 (1999), http://www.livingreviews.org/lrr-1999-2.
[16] R. Konoplya and A. Zhidenko, Rev. Mod. Phys. **83,** 793 (2011).
[17] E. Berti, V. Cardoso, and A. O. Starinets, Classical Quantum Gravity **26,** 163001 (2009).
[18] R. A. Konoplya and A. V. Zhidenko, Phys. Lett. B **609,** 377 (2005).
[19] W. Hu, R. Barkana, and A. Gruzinov, Phys. Rev. Lett. **85,** 1158 (2000).
[20] T. Matos and L. A. Urena-Lopez, Phys. Rev. D **63,** 063506 (2001).
[21] C. Boehmer and T. Harko, J. Cosmol. Astropart. Phys. 06 (2007) 025.
[22] M. Megevand, I. Olabarrieta, and L. Lehner, Classical Quantum Gravity **24,** 3235 (2007).