

General Linewidth Formula for Steady-State Multimode Lasing in Arbitrary Cavities

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A formula for the laser linewidth of arbitrary cavities in the multimode nonlinear regime is derived from a scattering analysis of the solutions to semiclassical laser theory. The theory generalizes previous treatments of the effects of gain and openness described by the Petermann factor. The linewidth is expressed using quantities based on the nonlinear scattering matrix, which can be computed from steady-state *ab initio* laser theory; unlike previous treatments, no passive cavity or phenomenological parameters are involved. We find that low cavity quality factor, combined with significant dielectric dispersion, can cause substantial deviations from the Schawlow-Townes-Petermann theory.

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The intrinsic linewidth of a laser arises from quantum fluctuations and would be zero in the absence of spontaneous emission. It is the most important property of lasers which arises from the quantization of the electromagnetic field. Its value depends on the properties of the specific laser cavity and gain medium, and was first calculated in the seminal work of Schawlow and Townes (ST), who found the famous linewidth formula [1]

$$\delta\omega_{\text{ST}} = \frac{\hbar\omega_0\gamma_c^2}{2P}, \quad (1)$$

where ω_0 is the frequency of the laser mode, γ_c is the linewidth of the relevant passive cavity resonance, and P is the modal output power. Note that in this formula the properties of the gain medium are absent.

Improved theoretical analyses over the next several decades found three multiplicative corrections to the ST formula, all of which tended to increase the linewidth, in some cases by large factors [2,3]. One correction factor arises from incomplete inversion of the gain medium, and a second one from indirect phase fluctuations due to the instantaneous intensity change caused by spontaneous emission (the Henry α factor) [3]. The third correction—possibly the most interesting and complicated one, and the main focus of this Letter—is the Petermann factor K . First discovered in the context of transverse gain-guided semiconductor lasers [4] and subsequently generalized [2,5–11], the Petermann factor arises from the non-Hermitian nature of the laser wave equation, due to the presence of the gain medium as well as the openness of the laser cavity (i.e., spatially nonuniform outcoupling loss). It always leads to an enhancement of the linewidth, even with uniform gain and no gain guiding. Typically, it is calculated from the nonorthogonal passive cavity resonances as

$$K = \left| \frac{\int dr |\varphi(r)|^2}{\int dr \varphi(r)^2} \right|^2, \quad (2)$$

where the integrals are taken over the cavity. In effect, the Petermann factor changes the ST linewidth by the replacement $\gamma_c^2 \rightarrow K\gamma_c^2$. This is a significant correction for lasers with large outcoupling; Ref. [2] measured it to be in the range 1.1–1.6 for conventional semiconductor lasers. We shall refer to the standard theory, inclusive of the Petermann factor, as the Schawlow-Townes-Petermann (STP) theory.

The extensive and impressive literature on the Petermann factor [2,4–11] has, with one major exception [9], only treated single-mode lasing near threshold, neglecting the effects of spatial hole-burning. And apart from a recent paper by Schomerus [11], the literature has exclusively treated one-dimensional or waveguide lasers and, thus, is not directly applicable to the variety of complex laser cavities developed during the past twenty years, such as microdisk and deformed-disk lasers, photonic crystal lasers, and random lasers. In this Letter, we derive a general formula for the intrinsic laser linewidth in arbitrary cavities, valid far from threshold, with strong spatial hole burning, and in the multimode regime. The formula relates the linewidth to a nonlinear self-consistent scattering matrix (S matrix), and is based on the recently developed steady state *ab initio* laser theory (SALT) [12–15].

SALT is a method for solving the steady-state lasing properties of arbitrary lasing structures without directly integrating the semiclassical laser equations. “Semiclassical” here refers to the fact that the field is treated via the classical Maxwell equations, whereas the properties of the gain medium are obtained from a quantum-mechanical calculation of a multilevel atom. SALT treats the openness of the cavity exactly, and the

nonlinear modal interactions and gain saturation are included to infinite order. Its results agree well with numerical integration of the laser equations, but it is computationally much more efficient [16,17]. It has been applied to complex laser structures such as random [14] and photonic crystal lasers [18]. We shall show that this method can be combined with the quantum input-output theory of Refs. [19,20], to calculate quantum fluctuation properties *ab initio*, in terms of quantities obtainable from SALT. SALT associates each laser mode with a scattering pole—an eigenstate of a classical nonlinear S matrix which has an infinite eigenvalue at a real frequency. We show that the linewidths of a multimode laser are determined by the residues of those poles. For a low- Q cavity with significant dielectric dispersion, the generalized linewidth formula predicts a substantial deviation from the STP behavior: the linewidth is significantly *less* than the standard Petermann correction predicts, and it has an anomalous power dependence.

The multimode SALT equations are [15]

$$\left[\nabla^2 + \left(\epsilon_c(\vec{r}) + \frac{\gamma_\perp D(\vec{r})}{\omega_\mu - \omega_a + i\gamma_\perp} \right) \omega_\mu^2 \right] \Psi_\mu(\vec{r}) = 0, \quad (3)$$

$$D(\vec{r}) = D_0(\vec{r}) \left[1 + \sum_{\nu=1}^n \Gamma_\nu |\Psi_\nu(\vec{r})|^2 \right]^{-1},$$

where Ψ_μ is the μ th steady-state lasing mode, ω_μ is its frequency, ϵ_c is the passive cavity dielectric function, γ_\perp is the gain medium linewidth, ω_a is the atomic transition frequency, $D_0(\vec{r})$ is the (possibly spatially varying) pump, and $\Gamma_\nu \equiv \gamma_\perp^2 / [\gamma_\perp^2 + (\omega_\nu - \omega_a)^2]$ is the gain curve. The effective pump $D(\vec{r})$ contains an infinite-order nonlinear “hole-burning” term, which gives rise to mode competition and gain saturation in a quantitatively precise manner. These coupled, time-independent, nonlinear equations are solved with the boundary condition of purely outgoing waves with frequency ω_μ at infinity; the solution algorithm is discussed in Refs. [14,15,17].

From the solution to (3), we can compute an effective self-consistent S matrix for any complex frequency ω , not just the discrete lasing frequencies ω_μ [21]. By definition, this S matrix has one or more poles on the real- ω axis, at $\omega = \omega_\mu$. The S matrix can now be used to study the effects of vacuum fluctuations and spontaneous emission. Let us suppose the cavity has scattering channels indexed by $j = 1, 2, \dots, N$ (the nature of these scattering channels depends on the scattering geometry; they could be waveguide modes or incoming and outgoing spherical waves, for example). The input and output photon operators are denoted by a_1, \dots, a_N and b_1, \dots, b_N respectively, and obey an “input-output” relation [22,23]:

$$b_i(\Omega) = \sum_j S_{ij}(\Omega) a_j(\Omega) + \sum_\rho V_{i\rho}(\Omega) d_\rho^\dagger(-\Omega). \quad (4)$$

Here the frequency Ω is measured from the classical lasing frequency of interest, $\Omega \equiv \omega - \omega_0$. The d_ρ 's are ladder operators for the external reservoirs corresponding to the gain medium, with the index ρ denoting appropriate degrees of freedom in the cavity or reservoir. Note that they enter with frequency $-\Omega$; this prescription will be needed to satisfy causality.

In order for a , b , and d to obey canonical commutation relations, e.g., $[a_i(\Omega), a_j(\Omega')] = \delta_{ij} \delta(\Omega - \Omega')$, the S matrix must be related to the reservoir coupling coefficients by the fluctuation-dissipation relation [22]

$$SS^\dagger - VV^\dagger = \mathbf{1}, \quad (5)$$

where $\mathbf{1}$ is the $N \times N$ identity matrix. Next, we define

$$a_j(t) = \frac{1}{\sqrt{2\pi}} \int d\Omega a_j(\Omega) e^{-i\Omega t} \quad (6)$$

and similarly for $b_j(t)$ and $d_\rho(t)$, describing quantum amplitudes for the field envelopes. Inserting into (4) gives

$$b_i(t) = \int dt' \left[\sum_k \int \frac{d\Omega}{2\pi} S_{ik}(\Omega) e^{-i\Omega(t-t')} \right] a_k(t') + \int dt' \left[\sum_\rho \int \frac{d\Omega}{2\pi} V_{i\rho}(\Omega) e^{-i\Omega(t-t')} \right] d_\rho^\dagger(t'). \quad (7)$$

The first term describes scattering of input photons, and the second describes emission from the gain medium.

S is strongly constrained by its symmetries. First, optical reciprocity [24] implies that S can be written as a symmetric matrix, so it has the eigenvalue decomposition

$$S = \sum_n |\psi_n\rangle \frac{s_n}{\langle \psi_n^* | \psi_n \rangle} \langle \psi_n^* |, \quad (8)$$

where each $|\psi_n\rangle$ denotes a right eigenvector of S with eigenvalue s_n , and $\langle \psi_n^* |$ denotes its unconjugated transpose. These eigenvectors are biorthogonal ($\langle \psi_m^* | \psi_n \rangle = 0$ for $m \neq n$) and power normalized ($\langle \psi_n | \psi_n \rangle = 1$).

Assume for convenience that the cavity dielectric function, ϵ , is real, and that it has a resonance near ω_0 . The S matrix will be unitary and for a high- Q cavity [25] one of its eigenvalues takes the approximate form:

$$s_0(\Omega) \approx e^{i\varphi(\Omega)} \frac{\Omega - i\gamma_c/2}{\Omega + i\gamma_c/2},$$

where φ is an irrelevant phase factor and γ_c is the cavity lifetime. Such an eigenvalue satisfies the conditions that for real Ω , it is unimodular, and the requirement of time-reversal symmetry, that the poles and zeros of the S matrix lie at conjugate positions in the complex Ω plane. Adding gain pushes the zero and pole up in the complex frequency plane. The eigenvalue takes the form

$$s_0(\Omega) \approx e^{i\varphi(\Omega)} \frac{\Omega - i\Gamma_z}{\Omega + i\Gamma_p}, \quad (9)$$

where Γ_z and Γ_p are the distances of the zero and pole from the real axis. As $\Gamma_p \rightarrow 0^-$, the pole approaches the real axis and the lasing threshold is reached; within the high- Q approximation the eigenvalue takes the form (9) with $\Gamma_z \approx \gamma_c$ (the zero moves up the same distance as the pole). This approximation leads directly to the ST formula (high Q will imply $K \approx 1$). For arbitrary Q , the S matrix near $\Omega = 0$ takes the form (9), with a generalized residue $\Gamma_L(\Omega)$ replacing $\Omega - i\Gamma_z$. Let us denote the eigenvector corresponding to this diverging eigenvalue by Ψ . In the S matrix decomposition (8), the term with s_0 dominates, so we can write

$$S \approx |\Psi\rangle \frac{s_0}{\langle \Psi^* | \Psi \rangle} \langle \Psi^*|. \quad (10)$$

Using this together with Eq. (5) gives

$$VV^\dagger \approx |\Psi\rangle \frac{1}{|\Psi^T \Psi|^2} \frac{|\Gamma_L|^2}{\Omega^2 + \Gamma_p^2} \langle \Psi|. \quad (11)$$

This equation is satisfied by the ansatz

$$V_{i\rho} = \frac{1}{\Psi_L^T \Psi_L} \frac{\Gamma_L}{\Omega + i\Gamma_p} \Psi_L^i u_\rho, \quad (12)$$

where u is some vector satisfying $\sum_\rho u_\rho^\dagger u_\rho = 1$, and Ψ_L^i is the i th component of the S -matrix eigenvector for the lasing mode. Note that this relation applies not just to the first lasing mode at threshold, but also for above-threshold steady-state lasing modes, using the self-consistent, nonlinear S matrix obtained from SALT.

Inserting (12) into (7) and performing the resulting contour integrations gives

$$b_i(t) = -\frac{\Gamma_L \Psi_L^i}{\Psi_L^T \Psi_L} \int^t dt' e^{-\Gamma_p(t-t')} F(t'), \quad (13)$$

$$F(t) \equiv \sum_j \Psi_{Lj} a_j(t) + i \sum_\rho u_\rho d_\rho^\dagger(t). \quad (14)$$

As expected, an output photon at time t is a superposition of incoming photon operators and reservoir operators from all *earlier* times.

Above threshold, the gain medium undergoes stimulated emission, and the laser field acquires a mean value. This can be described by adding a nonvanishing classical term ΔF alongside the input operator $F(t)$ in Eq. (13), which is modified to:

$$b_i(t) = B_i - \frac{\Gamma_L \Psi_L^i}{\Psi_L^T \Psi_L} \int^t dt' e^{-\Gamma_p(t-t')} F(t'), \quad (15)$$

where the complex number B_i is the steady-state classical outgoing field amplitude in channel i of the mode emitting at ω_0 . It is related to Ψ_L^i by

$$|B_i|^2 = \frac{P}{\hbar\omega_0} |\Psi_L^i|^2, \quad (16)$$

where P is the total output power of the mode.

Due to the fluctuation operator $F(t)$, the phase of the output field has a quantum uncertainty; the rate at which this uncertainty increases with time gives the laser coherence time scale. The fluctuation-induced phase changes are fed back into the classical value of B_i , causing a random drift in the phase of the laser field. We ignore this feedback, instead taking a fixed value for B_i for all t . This is justifiable because the integrand in (15) vanishes exponentially for $t' \lesssim -T$, where $T = 1/\Gamma_p$ will turn out to be the coherence time. The calculations below apply to times much shorter than T .

The global phase of the B_i 's is arbitrary, so we choose B_i to be real and positive for a specific channel i . To study quantum fluctuations of the phase, we introduce the Hermitian quadrature operator [26]

$$\theta_i = \frac{i(b_i^\dagger - b_i)}{2B_i}. \quad (17)$$

For small phase angles, this corresponds to the phase of the laser output in channel i . Using (14) and (15), we compute the quantity $\langle \theta_i(t_1)\theta_i(t_2) \rangle$, taking $\langle a \rangle = \langle d \rangle = \langle a_i^\dagger(t_1)a_j(t_2) \rangle = 0$ and taking the white noise correlator

$$\langle d_\rho(t_1)d_\nu^\dagger(t_2) \rangle = f_\rho \delta_{\rho\nu} \delta(t_1 - t_2), \quad (18)$$

where $f_\rho = [P_2/(P_2 - P_1)]_\rho$ describes the local population inversion [22]. The ‘‘zero-point’’ contributions to $\langle \theta_i(t_1)\theta_i(t_2) \rangle$ from the photon input and the gain medium cancel exactly, leaving

$$\langle \theta_i(t_1)\theta_i(t_2) \rangle = \frac{|\Gamma_L|^2}{|\Psi_L^T \Psi_L|^2} \frac{\hbar\omega_0}{4\Gamma_p P} e^{-\Gamma_p|t_1 - t_2|} \bar{f}, \quad (19)$$

where P is the modal output power from Eq. (16), and

$$\bar{f} \equiv \sum_\rho f_\rho |u_\rho|^2 \quad (20)$$

is the inversion factor correction mentioned at the beginning of this Letter.

The phase uncertainty accumulated over time Δt is $\langle [\theta_i(t + \Delta t) - \theta_i(t)]^2 \rangle = \Delta\omega \Delta t + O(\Delta t^2)$, where

$$\Delta\omega = \frac{|\Gamma_L|^2}{|\Psi_L^T \Psi_L|^2} \frac{\hbar\omega_0}{2P} \bar{f} \equiv \frac{\hbar\omega_0 \gamma_L^2}{2P} \bar{f}. \quad (21)$$

This is our central result: a general linewidth formula in which $|\Gamma_L|^2/|\Psi_L^T \Psi_L|^2 \equiv \gamma_L^2$ replaces the quantity $K\gamma_c^2$ in the conventional Schawlow-Townes-Petermann linewidth formula. We can think of γ_L as a generalized cavity decay rate, which has been corrected for the presence of gain,

openness, hole burning, and gain saturation. It is directly calculable, *ab initio* and with no phenomenological parameters, from the nonlinear classical S matrix of SALT. The lasing eigenvector is found by diagonalizing the S matrix, and the crucial quantity, Γ_L , is found by numerically integrating the eigenvalue around each lasing pole. This formula only includes the contribution to the laser linewidth from direct phase fluctuations; the indirect phase fluctuations [3] have been omitted for simplicity.

The relation of the Petermann factor to the residue of the lasing pole for a waveguide laser was emphasized early on by Henry [6], and developed for more general cavities in an S -matrix formulation by Schomerus *et al.* [27], but in both cases for a single lasing mode at threshold, i.e., without nonlinear effects. Goldberg, Milonni, and Sundaram [9] gave an excellent and detailed analysis of the linewidth for multimode lasing, including nonlinear effects, but using an approach only applicable to one-dimensional cavities with spatially uniform dielectric functions. To our knowledge,

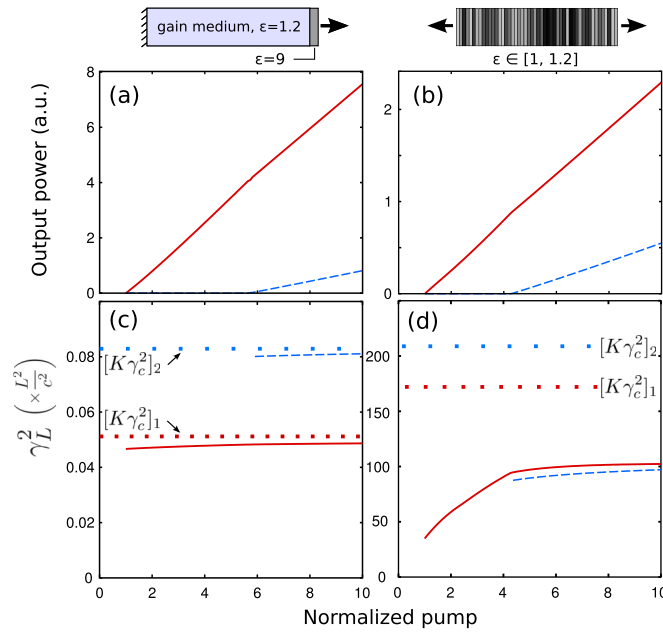


FIG. 1 (color online). Output power and cavity decay rates γ_L^2 for two uniformly pumped one-dimensional microcavity lasers. A slab of gain material with background $\epsilon = 1.2$, bounded on the left by a perfect mirror and on the right by an $\epsilon = 9$ slab (5% of the total length) acting as a partially transmitting mirror (left schematic). A random laser consisting of 50 slabs of gain material, each with background ϵ uniformly distributed in $[1, 1.2]$ (right schematic). Both systems exhibit two-mode lasing at the high end of the pump range. Plots (a) and (b) show modal output powers vs the normalized pump [15]. Plots (c) and (d) show the square of the generalized cavity decay rate $\gamma_L^2 \equiv |\Gamma_L|^2/|\Psi_L^T \Psi_L|^2$ which determines the linewidths according to Eq. (21). Solid and dashed curves denote modes 1 and 2, respectively. The horizontal dotted lines show the conventional result, $K\gamma_c^2$, computed from the passive cavity quasimodes, which fails for the random laser.

our Eq. (21), combined with SALT, is unique in providing a quantitative method for calculating the intrinsic laser linewidth in arbitrary cavities and pump profiles in the multimode, nonlinear regime. Assuming steady-state multimode lasing exists, the present theory makes no significant further approximations, and hence it can be used to evaluate the validity of the Schawlow-Townes-Petermann linewidth formula [2].

We can connect Eq. (21) to previous results involving quasimodes, such as Refs. [4,5,7], by examining the S matrix of a *passive* cavity. A quasimode $\varphi(r)$ is a purely outgoing solution to the wave equation for a passive cavity with dielectric function $\epsilon(r)$, at complex frequency ω_p , where $\text{Im}(\omega_p) \equiv -\gamma_c/2$. Let Ψ be the S -matrix eigenvector for this pole, normalized so that $\Psi^\dagger \Psi = 1$, and let Γ be the residue of the eigenvalue. It can be shown that [21]

$$\text{Im} \left[\omega_p^2 \int dr \epsilon(r) |\varphi(r)|^2 \right] = -\text{Re}[\omega_p], \quad (22)$$

$$\int \epsilon(r) \varphi^2(r) = \left[\frac{i}{\Gamma} - \frac{i}{2\omega_p} \right] \Psi^T \Psi. \quad (23)$$

Here the spatial integrals are taken over the cavity. For real $\epsilon(r)$, and in the limit $|k_p| \gg \Gamma \sim \gamma_c$, (22) and (23) give

$$\frac{|\Gamma|^2}{|\Psi^T \Psi|^2} \approx \left| \frac{\int dr \epsilon(r) |\varphi(r)|^2}{\int dr \epsilon(r) \varphi(r)^2} \right|^2 \gamma_c^2 = K \gamma_c^2. \quad (24)$$

Thus, the conventional Petermann factor times γ_c^2 is approximately equal to a quantity similar to γ_L^2 , except that it involves the eigenvalue residue and eigenvector of the passive cavity S matrix. Note that both (24) and its active cavity generalization in Ref. [11] do not include the effects of dielectric dispersion, which can have a significant effect on γ_L .

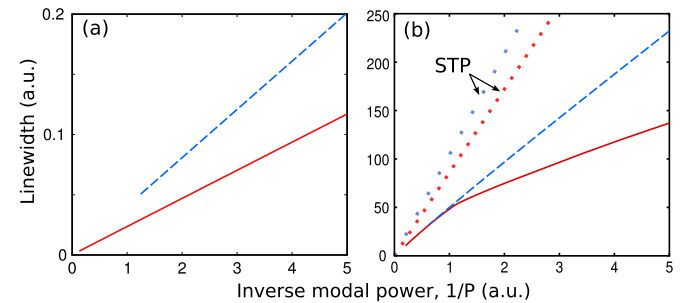


FIG. 2 (color online). Laser linewidth vs inverse modal output power $1/P$, for the two lasers studied in Fig. 1. The linewidths are computed using Eq. (21), assuming the inversion factor $\bar{f} = 1$. (a) The high- Q cavity laser linewidths show the standard $1/P$ dependence for both modes. (b) The linewidth of the first mode of the random laser deviates strongly from the $1/P$ Schawlow-Townes-Petermann dependence at lower pump values. At large pump values the linewidths of both mode 1 (solid curve) and mode 2 (dashed curve) vary as $1/P$, but with values roughly half that of the standard STP prediction (dotted curve).

Figure 1 compares γ_L^2 to $K\gamma_c^2$ for two one-dimensional microcavity multimode lasers: a high- Q , uniform cavity for which the two quantities agree rather well, and a low- Q random laser, for which major deviations are found. In particular, for the random laser, at pump strengths up to four times threshold, γ_L for the first lasing mode depends strongly on P , causing the overall power dependence to depart substantially from the standard $1/P$ dependence (Fig. 2). For higher pump strengths, γ_L is approximately constant, but the conventional linewidth prefactor $K\gamma_c^2$ *overestimates* it by almost a factor of 2. In the standard theory, the STP linewidth is a lower bound set by field quantization, but insofar as it relies on passive cavity quantities it is not a reliable bound. Analysis of our results indicates that the deviation from the STP theory arises from low cavity Q and from the frequency dispersion of the dielectric constant of the gain medium, which significantly reduces the residue of the lasing pole at threshold compared to the passive cavity. We do not believe that the apparent violation of the STP bound indicates any new quantum fluctuation properties. In future work, our generalized linewidth formula will allow such issues to be studied systematically.

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