Mach-Number-Invariant Mean-Velocity Profile of Compressible Turbulent Boundary Layers

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A series of Mach-number- (M) invariant scalings is derived for compressible turbulent boundary layers (CTBLs), leading to a viscosity weighted transformation for the mean-velocity profile that is superior to van Driest transformation. The theory is validated by direct numerical simulation of spatially developing CTBLs with M up to 6. A boundary layer edge is introduced to compare different M flows and is shown to better present the M-invariant multilayer structure of CTBLs. The new scalings derived from the kinetic energy balance substantiate Morkovin's hypothesis and promise accurate prediction of the mean profiles of CTBLs.

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The compressible turbulent boundary layer (CTBL) has received considerable attention because of its relevance to high speed aircraft, flow over blades, etc. Compared to the incompressible turbulent boundary layer (ITBL), the CTBL is less clear, owing to the compressibility and thermodynamic effects; consequently, it presents a tremendous challenge to theory, modeling, and computation [1,2]. Efforts have long been devoted to searching for transformations or scalings to reduce the mean-field distribution of the CTBL to its incompressible form. A moderate success was achieved for the mean-velocity profile (MVP) via van Driest transformation (VDT) [3] and for the Reynolds stress via Morkovin's scaling [1,4]. These Mach-number- (M) invariant forms serve as important standards to validate turbulence models and numerical simulations of the CTBL [1,5,6]. This Letter proposes a new scaling for mapping the MVP with sound physics and higher accuracy.

VDT, expressed by $\bar{u}_{vD}^{+} = \int_0^{\bar{u}^+} \sqrt{\bar{\rho}^+} d\bar{u}^+$ where the plus superscript denotes wall unit normalization [2], is the most important relation in compressible turbulent wall-bounded flows. It retains the log law of incompressible flows, i.e., $\bar{u}_{vD}^{+} = \kappa^{-1} \ln y^+ + B$, where $\kappa \approx 0.41$ and $B \approx 5.2$ for the quasiadiabatic wall condition [7]. VDT can be derived under two assumptions: (1) $-(\bar{\rho}\,\overline{u'v'})^+$ is invariant to M; and (2) Prandtl's mixing length $\ell_m^+ = (\overline{-u'v'})^{+1/2}/(\partial \bar{u}/\partial y)^+$ is also M invariant [5,6], and thus Prandtl's proposal of $\ell_m^+ = \kappa y^+$ holds in the log layer [3]. The two assumptions give rise to the M invariance of $\sqrt{\bar{\rho}^+}\partial \bar{u}^+/\partial y^+$ and further to VDT. The first assumption has been confirmed and referred to as Morkovin's scaling [4,8]. The second is not thoroughly studied [6], however. Over the past decades, a few experiments showed that ℓ_m^+ had an observable, although not strong, M effect, particularly in the outer part of the boundary layer [5,9]. Further evidence comes from direct numerical simulation (DNS) data [10], which show that the van Driest transformed MVP does not collapse well in the wake region, invalidating the M invariance of $\sqrt{\bar{\rho}^+}\partial \bar{u}^+/\partial y^+$ and thus the second assumption.

After van Driest, Maise *et al.* [9] presented a transformation for the wake region, and Huang *et al.* [7] proposed a composite transformation by considering the near wall variation of ℓ_m^+. Nevertheless, no existing transformation can cope with the whole MVP. The present work finds an M-invariant mixing length to resolve this problem. The analysis is based on DNS of spatially developing, zero-pressure-gradient, flat plate CTBLs in the quasiadiabatic isothermal wall condition, whose computational details are explained in Ref. [11].

The CTBLs have two control parameters, Reynolds number (Re) and M; in order to compare different M flows, the first important issue is to define an edge of the CTBL, and thus a Re. There is no widely accepted definition of this edge. For example, Lagha *et al.* [12] used the δ_{99} defined by $\bar{u}(y = \delta_{99}) = 99\% \bar{u}_\infty$ while Guarini *et al.* [13] used van Driest transformed δ_{99}^{vD} defined by $\bar{u}_{vD}(y = \delta_{99}^{vD}) = 99\% \bar{u}_\infty^{vD}$.

We suggest an extension of Townsend's structure parameter, defined as the ratio of the Reynolds shear stress to the turbulent kinetic energy [1], to define a boundary layer thickness. The extension goes to all the components of the Reynolds stress tensor: $a_{u_iu_j} = |\overline{u_i'u_j'}/\overline{u_k'u_k'}|$, where $a_{u_iu_j}$ are named Reynolds stress structure parameters (RSSPs). Morkovin's scaling asserts that RSSPs are invariant to M at the same Re. We argue that RSSPs can be traced to coherent vortices in a boundary layer, thus displaying multilayer behavior. Therefore, a suitable CTBL thickness can be located through identifying the M-invariant multilayer features of RSSPs.

In Fig. 1(a), the RSSPs for different M are presented using Re_τ with the wall-normal coordinate scaled by δ_{99}, following convention. In the bulk region, one finds quasi-constant $a_{u_iu_j}$ [1,8], but farther away from the wall, the M effect is significant. We argue that the observed M effect is due to improper δ_{99} and Re_τ. In Fig. 1(b), the RSSPs are replotted using $\mathrm{Re}_{\delta_{vw}}$ where δ_{vw} is the thickness of $a_{vv} = a_{ww}$. Comparing Figs. 1(a) and 1(b), one finds that the M-invariant region extends to $\delta_{vu} \approx 1.2\delta_{vw}$, significantly above δ_{99}, where δ_{vu} is the thickness of $a_{vv} = a_{uu}$ (above δ_{vu}, an apparent M effect appears, possibly because the intermittency there is influenced by the Mach cone of flow disturbances [1,8]). Note that the significance of δ_{vw} over δ_{99} becomes more obvious if one notices that it is not the only M-invariant thickness, but so are several other layer thicknesses, including the viscous sublayer, the buffer layer, the bulk flow region, and the entrainment or intermittent layer, which are made up of an M-invariant four-layer structure that we believe is a general feature of the CTBL.

The validity of δ_{vw} as a suitable thickness for the CTBL is further confirmed by calculating the skewness and flatness factors. In previous studies, they were found to vary with M in the entrainment or intermittent layer [1,8]. In our plotting, however, they are M invariant up to δ_{vu} at a given $\mathrm{Re}_{\delta_{vw}}$, agreeing with Morkovin's scaling. This agreement implies that the previously observed M dependence is at least partially due to the inappropriate boundary layer thickness and Re used. Hereinafter, δ_{vw} and $\mathrm{Re}_{\delta_{vw}}$ are applied to compare different M flows.

We now discuss the form of the M-invariant mixing length. The turbulent kinetic energy (TKE) equation [14] can be written as $C = P + T + \Pi + D + M - \phi$, where C, P, T, Π, D, M, and ϕ represent, respectively, advection, production, transport, pressure dilatation, viscous diffusion, mass flux associated with density fluctuations, and viscous dissipation. P almost entirely comes from $-\bar{\rho}\,\widetilde{u''v''}(\partial\tilde{u}/\partial y)$, where the tilde denotes the Favre average defined by $\tilde{f} = \overline{\rho f}/\bar{\rho}$ and the double prime denotes Favre fluctuation. The ϕ, at moderate M, can be reduced to $\bar{\mu}\,\overline{\omega_i'\omega_i'}$ [14]. Above the buffer layer and beneath the uniform flow, P and ϕ are in quasibalance [1] and the ratio $\Theta \equiv P/\phi$ negligibly depends on M according to our DNS data. In the literature, $\overline{\omega_i'\omega_i'}^+$ is also found to be independent of M [12]. As a result, the M invariance of $\overline{\omega_i'\omega_i'}^+$, $-\bar{\rho}^+\widetilde{u''v''}^+$ (denoted by W hereinafter), and Θ yield an important finding that $(\partial\tilde{u}/\partial y)^+/\bar{\mu}^+$, instead of $\sqrt{\bar{\rho}^+}\partial\bar{u}^+/\partial y^+$, is M invariant. A direct consequence is that the M-invariant scaling of Prandtl's mixing length is $\ell_{MI}^+ = (\sqrt{\bar{\rho}^+}\bar{\mu}^+)\ell_m^+$, where ℓ_m^+ is written in the Favre form: $\ell_m^+ = (-\widetilde{u''v''})^{+1/2}/(\partial\tilde{u}^+/\partial y^+)$.

We briefly discuss the physical bases of the findings. It is long known that compressibility effects owing to dilatation and fluctuation of thermodynamic quantities are negligibly small in wall-bounded turbulence, at least at moderate M [8]. The ϕ is reduced to $\bar{\mu}\,\overline{\omega_i'\omega_i'}$ based on this fact [14]. The $\overline{\omega_i'\omega_i'}^+$, i.e., enstrophy, is mainly generated by vortex stretching as in incompressible turbulence [15], thus displaying M invariance. The W, the Reynolds shear stress in the Favre form, actually has a better M-invariant property than the conventional Reynolds stress, possibly owing to the fact that Favre averaging, as is well known, best demonstrates the similarity between the compressible and incompressible Navier-Stokes equations [6]. And finally, P and ϕ being in quasibalance in the bulk of a boundary layer [1], Θ is insensitive to M. The rationale for M invariance of $(\partial\tilde{u}/\partial y)^+/\bar{\mu}^+$ can be explained as follows. The wall-normalized viscosity away from the wall decreases with increasing M, thus reducing energy dissipation, and consequently, less slowing down the fluid. This gentler $(\partial\tilde{u}/\partial y)^+$ can be restored to the ITBL one by $(\partial\tilde{u}/\partial y)^+/\bar{\mu}^+$ (denoted by S hereinafter).

The above arguments lead further to a series of M-invariant scalings for flow quantities such as the eddy viscosity coefficient ν_t^+ [5], the Reynolds stress dissipation length L_{uv}^+ [5], the Kolmogorov length η^+ [1], and the TKE budgets B_{TKE}^+ [14]. Specifically, the M-invariant

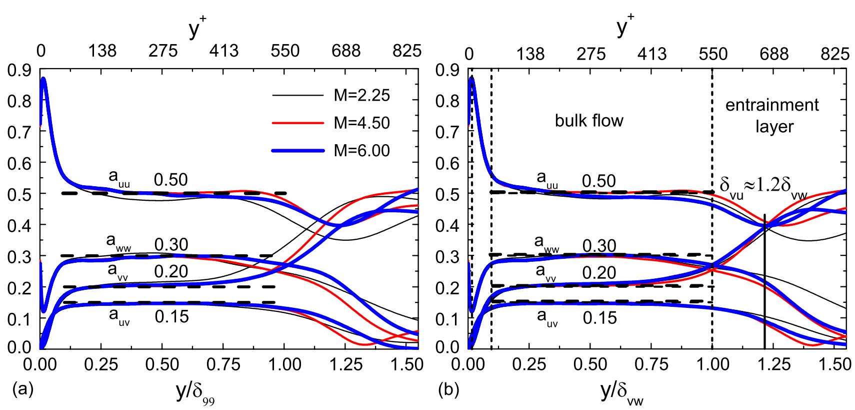

FIG. 1 (color online). Wall-normal distributions of $a_{u_iu_j} = |\overline{u_i'u_j'}/\overline{u_k'u_k'}|$ versus (a) y/δ_{99} at $\mathrm{Re}_\tau = \bar{\rho}_w\bar{u}_\tau\delta_{99}/\bar{\mu}_w = 575$ and (b) y/δ_{vw} at $\mathrm{Re}_{\delta_{vw}} = \bar{\rho}_w\bar{u}_\tau\delta_{vw}/\bar{\mu}_w = 550$.

scalings of the above quantities are $(\bar{\rho}^+\bar{\mu}^+)\nu_t^+$, $(\sqrt{\bar{\rho}^+}\bar{\mu}^+)L_{uv}^+$, $\sqrt{\bar{\rho}^+/\bar{\mu}^+}\eta^+$, and $B_{TKE}^+/\bar{\mu}^+$, respectively. Morkovin's hypothesis claimed that any difference (of the large scale motion) between CTBL and ITBL could be removed by incorporating the wall-normal variations of $\bar{\rho}$, $\bar{\mu}$, and thermal conductivity $\bar{k}$. The above M-invariant scalings specify the otherwise "vague" Morkovin's hypothesis by extending to quantities other than Reynolds stress, and by incorporating both $\bar{\rho}$ and $\bar{\mu}$ (note $\bar{k} \propto \bar{\mu}$). In addition, the M-invariant scaling of $\eta_{MI}^+ \equiv (\sqrt{\bar{\rho}^+/\bar{\mu}^+})\eta^+$ suggests that Morkovin's hypothesis also applies to small scale motion, extending his original claim. As shown in Fig. 2, η_{MI}^+ is remarkably M invariant throughout the boundary layer while η^+ in the inset is strongly influenced by M owing to the apparent M dependence of $\bar{\rho}^+$ and $\bar{\mu}^+$. Note that, considering the definition of η_{MI}^+, Fig. 2 also confirms the M invariance of $\overline{\omega_i'\omega_i'}^+$ and $\phi^+/\bar{\mu}^+$.

Figure 3 shows that ℓ_{MI}^+ depends little on M across the boundary layer. Above the buffer layer, the M invariance of ℓ_{MI}^+ is promised by the above derivation. In the near wall region, both S and W have a slight M effect, which is due to the large gradients of the mean fluid properties there. The M effects of S and W are similar, resulting in the M invariance of ℓ_{MI}^+ in the near wall region. Comparing ℓ_m^+ and ℓ_{MI}^+, the scaling coefficient $\sqrt{\bar{\rho}^+}\bar{\mu}^+ \approx (\bar{\rho}^+)^{-0.26}$ [1] is clearly less than one at large M and y/δ_{vw}, so as to remove the slight increase of ℓ_m^+ with increasing M observed previously [9]. For a $M = 6$, $\mathrm{Re}_{\delta_{vw}} = 550$ CTBL, $\sqrt{\bar{\rho}^+}\bar{\mu}^+$ is about 0.8 at δ_{vw}. It does not have a significant effect in the log coordinate (Fig. 3) but affects the MVP after integration.

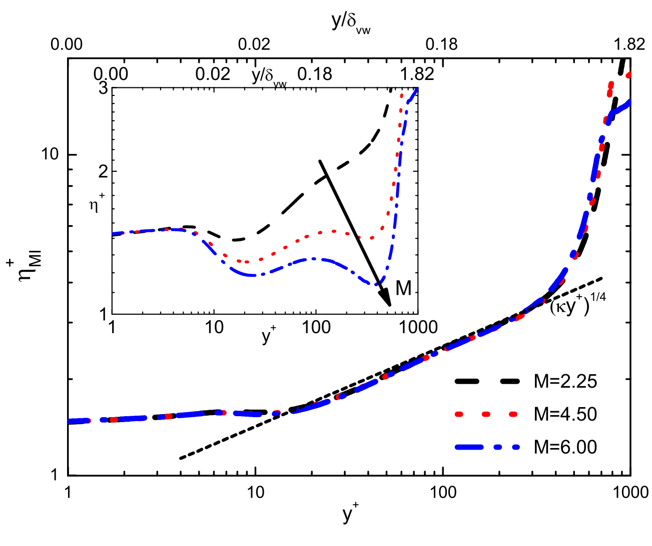

FIG. 2 (color online). M invariance of the density viscosity rescaled Kolmogorov length $\eta_{MI}^+ = (\sqrt{\bar{\rho}^+/\bar{\mu}^+})\eta^+$ at $\mathrm{Re}_{\delta_{vw}} = 550$, where $\eta^+ = (\bar{\mu}^{+3}/\phi^+)^{1/4}/\sqrt{\bar{\rho}^+}$ (inset) is the normal Kolmogorov length and $\eta_{MI}^+ = (\phi^+/\bar{\mu}^+)^{-1/4} \approx (\overline{\omega_i'\omega_i'}^+)^{-1/4}$. The dashed straight line shows $\eta_{MI}^+ = (\kappa y^+)^{1/4}$ in the log layer.

Now we are able to derive a transformation to map the whole MVP of the CTBL to that of the ITBL. The M invariance of S implies a viscosity weighted transformation: $u_{MI}^+ = \int_0^{\tilde{u}^+} \frac{1}{\bar{\mu}^+} d\tilde{u}^+$. Since S has a slight M effect below the log layer, as mentioned above, a refined transformation is derived in the following. For a CTBL, the wall-normal integrated mean momentum equation [14] in terms of ℓ_{MI}^+ reads $\ell_{MI}^+ = \sqrt{\bar{\tau}^+ - \bar{\mu}^{+2}S}/S$, where $\bar{\tau}^+$ is the total shear stress. Using the global M invariance of ℓ_{MI}^+, a refined transformation is obtained as

$$u_{MI}^+ = \int_0^{\tilde{u}^+} \frac{g}{\bar{\mu}^+} d\tilde{u}^+;$$

$$g = \frac{-\frac{S}{2} + \sqrt{(\frac{S}{2})^2 + (1 - \bar{\mu}^{+2}S)}}{(1 - \bar{\mu}^{+2}S)}. \tag{1}$$

Note $\bar{\tau}^+$ is set to one in Eq. (1). The rationale is as follows: in the constant stress layer, $\bar{\tau}^+ \approx 1$; above the constant stress layer, $g \approx 1$ independent of $\bar{\tau}^+$, which agrees with the M invariance of S.

Figure 4 compares the refined viscosity weighted transformation with VDT. The whole MVP of the CTBL collapses to the empirical MVP of the ITBL [Eq. (4) in Ref. [14]] with negligible M effect. In contrast, the van Driest transformed MVPs increase with increasing M, especially in the wake region (lower right inset). In essence, the improvement is attributed to the correct quantification of the dependence of ℓ_m^+ on the variation of the mean fluid properties. In the log layer, the scaling coefficient of ℓ_m^+, i.e., $\sqrt{\bar{\rho}^+}\bar{\mu}^+$, is very close to one, which explains the good performance of VDT there. Farther away from the wall, $\sqrt{\bar{\rho}^+}\bar{\mu}^+$ deviates from one and the

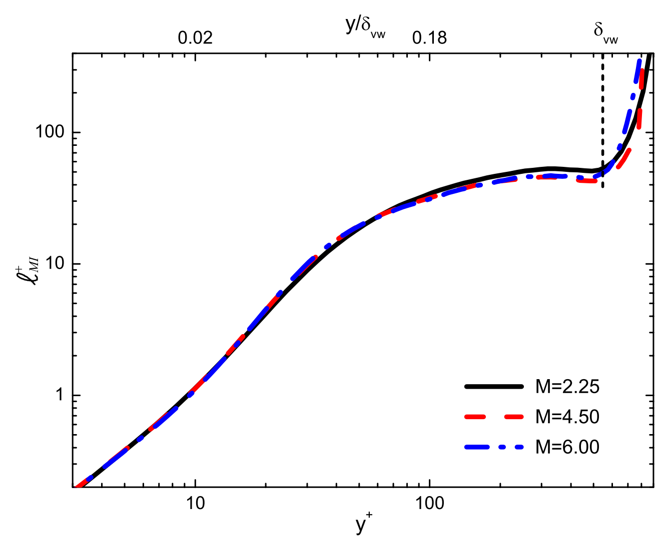

FIG. 3 (color online). Global M invariance of density viscosity rescaled Prandtl's mixing length $\ell_{MI}^+ = \sqrt{\bar{\rho}^+}\bar{\mu}^+\ell_m^+$ at $\mathrm{Re}_{\delta_{vw}} = 550$.

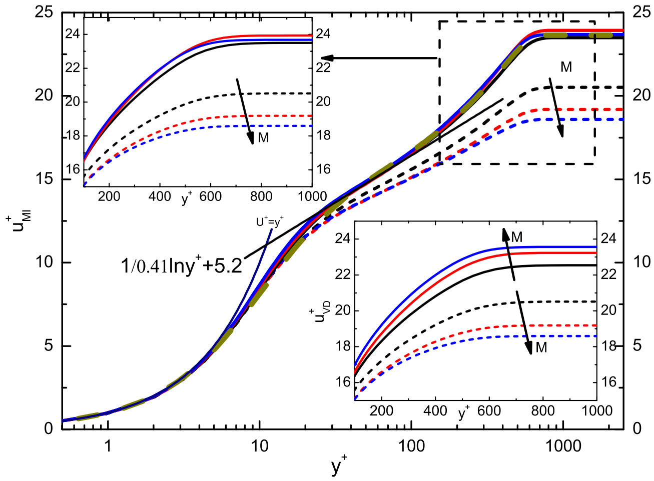

FIG. 4 (color online). MVPs of the CTBL transformed by the refined viscosity weighted transformation (solid lines) compared to the untransformed MVPs (short-dashed lines) and the empirical MVP of the ITBL [14] (long-dashed dark yellow line) at $Re_{\delta_{vw}} = 550$. The insets are the enlargement in the wake region. VDT is shown in the lower right inset, and the refined viscosity weighted transformation is shown in the upper left. The arrows indicate that M increases from 2.25 to 4.5 and to 6. See Fig. 3 for the color legend.

deviation increases with M, leading to a noticeable modification to VDT after integration.

A discussion is in order. The success of δ_{vw} indicates the validity of the concept of the "order function" (here $a_{u_iu_j}$) in identifying the multilayer structure in wall turbulence, which has been well tested in the study of incompressible channel or pipe and boundary layer flows [16–18]. These studies suggest that the multilayer structure is the result of symmetry breaking in the mean field induced by turbulent fluctuations and the order functions capture the transitions between different layer states. Here, we show that the multilayer structure exists also in the compressible turbulent boundary layer and is independent of M when revealed by $a_{u_iu_j}$.

The new M-invariant scalings and transformation quantify the influence of the variation of the mean fluid properties on the turbulence in CTBLs, and this quantification is critical for validating turbulence theories and computations. In addition, a prediction of the mean profiles of the CTBL is possible now. Specifically, the mean profiles of density, velocity, and temperature of CTBLs can be predicted by combining the inverse transformation of Eq. (1), a velocity-temperature (hence viscosity) relation such as Walz's equation [14], and the MVP of the ITBL [14]. Validation of the new proposals in other flows, at higher Re and M, with a pressure gradient, and for a nonadiabatic wall, needs further study. This is hopeful since our preliminary investigation indicates that the concepts of both multilayer structure and order function are valid in general in the presence of these mentioned complexities.

In summary, the M effect in the CTBL should be discussed with an appropriately defined $Re_{\delta_{vw}}$. The δ_{vw} best presents the M-invariant multilayer structure of CTBLs. We bolster Morkovin's hypothesis by deriving a series of new M-invariant scalings. The M-invariant mixing length leads to a viscosity weighted transformation for mapping the MVP of the CTBL with negligible M effect throughout the boundary layer. We hope these findings will improve the validation of turbulence theories and computations of the CTBL, provide more insights into the interactions between the velocity and thermal fields, and finally, promise accurate prediction of the mean profiles of the CTBL.

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