Generalized Monogamy of Contextual Inequalities from the No-Disturbance Principle

Ravishankar Ramanathan,[1] Akihito Soeda,[1] Paweł Kurzyński,[1,2] and Dagomir Kaszlikowski[1,3,*]

[1]*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117543 Singapore, Singapore*
[2]*Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland*
[3]*Department of Physics, National University of Singapore, 2 Science Drive 3, 117542 Singapore, Singapore*

In this Letter, we demonstrate that the property of monogamy of Bell violations seen for no-signaling correlations in composite systems can be generalized to the monogamy of contextuality in single systems obeying the Gleason property of no disturbance. We show how one can construct monogamies for contextual inequalities by using the graph-theoretic technique of vertex decomposition of a graph representing a set of measurements into subgraphs of suitable independence numbers that themselves admit a joint probability distribution. After establishing that all the subgraphs that are chordal graphs admit a joint probability distribution, we formulate a precise graph-theoretic condition that gives rise to the monogamy of contextuality. We also show how such monogamies arise within quantum theory for a single four-dimensional system and interpret violation of these relations in terms of a violation of causality. These monogamies can be tested with current experimental techniques.

Introduction.—The essence of the classical description of nature is the assumption that the physical world exists independently of any observers, and that the act of observation does not disturb it. This feature, called realism, was first brought to mainstream physics by Einstein, Podolsky, and Rosen in [1]. A mathematical consequence of realism is that there exists a joint probability distribution for the outcomes of measurements for all physical properties of the system [2]. It was shown by Bell [3] and by Kochen and Specker [4] that the property of realism is not present in quantum theory. Therefore, it is reasonable to treat realism as a hallmark of classicality and the lack of it as an indicator of quantumness. Clauser *et al.* [5] and Klyachko *et al.* [6] proposed minimal experimental tests of this feature by means of the Clauser-Horne-Shimony-Holt inequality and the Klyachko-Can-Binicoglu-Shumovsky (KCBS) inequality, respectively. These inequalities must be satisfied in any theory that incorporates realism, and their violations have been observed experimentally [7,8], confirming that nature is not compatible with realism.

Although Bell inequalities are studied as indicators of "local realism" and KCBS inequalities are treated as indicators of "noncontextuality," they have the same root, namely, the assumption of realism. Local realism is a special kind of realism where the additional constraint of locality is imposed, namely, that measurements in spatially separated systems do not influence each other. Bell inequalities thus require at least two correlated and spatially separated subsystems. The notion of noncontextuality, on the other hand, applies to a single system, and stipulates that the outcomes of any measurement are independent of any other measurement that can be jointly performed with it. Violation of the KCBS inequality implies that the system does not admit a noncontextual description for these

measurements. Contextual inequalities being applicable to single systems are arguably more fundamental than Bell inequalities in studying the quantumness of physical systems.

Quantum correlations as captured by the violation of Bell inequalities have been shown to obey the interesting property of monogamy [9]; if Alice is able to violate a Bell inequality with Bob, she is unable to violate the same Bell inequality with Charlie. This property arises only under certain conditions: namely, (i) Alice uses the same settings to violate Bell inequalities with both Bob and Charlie; (ii) No communication between Alice, Bob, and Charlie is allowed; (iii) Bob and Charlie cannot use more than two measurement settings; (iv) Alice tries to violate the very same Bell inequality with both Bob and Charlie. Bell monogamies are useful in secure quantum key distribution [10], interactive proof systems [11], and in the emergence of a local realistic description for correlations in the macroscopic domain [12].

The fact that the origin of Bell inequalities and contextual inequalities is the existence of joint probability distributions suggests that a similar monogamy relation may hold for contextual inequalities as well. Bell monogamy arises as a consequence of the no-signaling principle, which states that the probabilities of outcomes of measurement in one subsystem are independent of the choice of measurement in a spatially separated subsystem. An interesting question is how the properties of no signaling and monogamy translate to contextual inequalities.

In this Letter, we focus on KCBS-type contextual inequalities and show that there is a form of monogamy of their violations analogous to the monogamy of Bell inequality violations. We exploit the Gleason principle of no disturbance that is a generalization of the principle of no

signaling. The paper is organized as follows. We first formulate mathematically the principle of no disturbance. We then derive a monogamy relation for the simplest contextual inequality, namely, the KCBS inequality for five measurements, and show that it applies to any theory that obeys the principle of no disturbance, in particular quantum mechanics. Using techniques of graph theory, we show how these monogamies can be generalized to any contextual and Bell inequalities. In particular, we establish a proposition that any chordal graph representing a set of measurements admits a joint probability distribution and use this to identify the necessary and sufficient condition for a set of measurements to exhibit the monogamy of contextuality.

Principle of no disturbance.—To formulate the principle of no disturbance mathematically, let us consider a physical system on which one can perform several different measurements A, B, C, etc. Let us assume that measurements A and B can be jointly performed as can measurements A and C. This implies the existence of the joint probabilities $p(A = a, B = b)$ and $p(A = a, C = c)$ (where the small letters denote outcomes of the corresponding measurements). The principle of no disturbance is then the condition that the marginal probability $p(A = a)$ calculated from $p(A = a, B = b)$ is the same as that calculated from $p(A = a, C = c)$; i.e.,

$$\sum_b p(A = a, B = b) = \sum_c p(A = a, C = c) = p(A = a).$$

(1)

This property has been referred to as the Gleason property in [13], since it is the condition underlying Gleason's theorem. Note that when measurements B and C are performed on spatially separated systems, the principle of no disturbance reduces to that of no signaling. From here on we use $p(A = a, B = b)$ and $p(a, b)$ interchangeably wherever there is no possibility of confusion.

Monogamy of KCBS-type inequalities.—We concentrate first on the KCBS inequality from [6] that was introduced to test the quantumness of a single (three-level) system and construct a monogamy relation for it. Similar monogamies hold for any inequalities of this kind [14]. The KCBS inequality reads

$$\sum_{i=1}^{5} p(A_i = 1) \leq 2,$$

(2)

where A_i represents the measurements with outcomes $a_i = 0, 1$. These measurements are cyclically compatible [i.e., it is possible to experimentally determine $p(a_i, a_{i+1})$ (where one identifies a_6 with a_1)] and exclusive (i.e., $a_i a_{i+1} = 0$). Measurements A_{i-1} and A_{i+1} are said to provide two different contexts for the measurement A_i. These measurements can be represented by the "commutation graph" corresponding to a pentagon where the vertices of the pentagon graph represent the five measurements and the edges between any two vertices indicate that the two corresponding measurements can be jointly performed and

are mutually exclusive. The bound of 2 is derived under the assumption of existence of the joint probability distribution $p(a_1, a_2, \ldots, a_5)$. In graph theoretic terms, this bound corresponds to the independence number of the pentagon graph, which is the maximum number of mutually disconnected vertices in the graph. This inequality is violated in any contextual theory such as quantum theory where such a joint probability distribution does not exist. The KCBS inequality Eq. (2) is the necessary and sufficient condition for the existence of noncontextual description for these five measurements. Analogous inequalities can be constructed for a larger number of measurements as well [13,14].

We can now precisely state the definition of contextual monogamy as follows. A set of measurements is said to have "monogamous contextuality" if it can be partitioned into disjoint subsets, each of which can by themselves reveal contextuality, but which cannot all simultaneously be contextual. Let us derive a monogamy relation for the KCBS contextual inequality from the no-disturbance principle, along similar lines to the monogamy of Bell inequality violations derived from the no-signaling principle [9]. Consider two sets of cyclically compatible and exclusive measurements $\{A_i\}$ and $\{A_i'\}$. Each set gives rise to a KCBS inequality Eq. (2). Let us assume that the triple A_1, A_1', A_2' are jointly measurable and mutually exclusive, as is also the triple A_4, A_5, A_5'. This scenario is represented by the commutation graph in Fig. 1. Therefore, in addition to $p(a_i, a_{i+1})$ and $p(a_i', a_{i+1}')$, one can experimentally determine probabilities $p(a_1, a_1', a_2')$ and $p(a_5', a_4, a_5)$. This condition is similar to that imposed in the derivation of Bell monogamies, namely, that a common observer chooses the same settings for the violation of Bell inequalities with all other observers.

We introduce the no-disturbance principle (1) by setting $p(A_1 = 1) = p$ and $p(A_5' = 1) = q$. Mutual exclusiveness implies that $p(A_1' = 1) + p(A_2' = 1) \leq 1 - p$ and $p(A_4 = 1) + p(A_5 = 1) \leq 1 - q$ in addition to $p(A_i = 1) + p(A_{i+1} = 1) \leq 1$ and $p(A_i' = 1) + p(A_{i+1}' = 1) \leq 1$. However, this already implies $\sum_{i=1}^{5} p(A_i = 1) \leq 2 - q + p$ and $\sum_{i=1}^{5} p(A_i' = 1) \leq 2 - p + q$ and therefore the monogamy relation
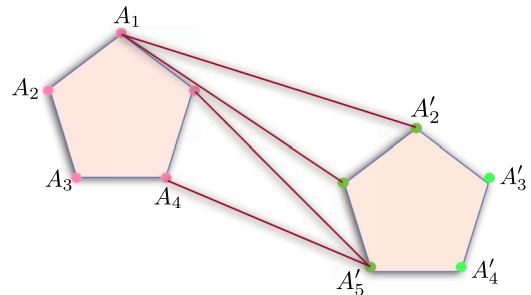

FIG. 1 (color online). Graphical representation of two KCBS inequalities that satisfies the monogamy relation.

$$\sum_{i=1}^{5} p(A_i = 1) + \sum_{i=1}^{5} p(A_i' = 1) \le 4 \qquad (3)$$

holds. Therefore, only one KCBS inequality out of the two sets $\{A_i\}$ and $\{A_i'\}$ can be violated in all theories that obey the no-disturbance principle such as quantum mechanics.

If, however, the principle of no disturbance does not hold, it is possible to violate both inequalities simultaneously. This is because in this case $p(a_1)$ calculated from $p(a_1, a_1', a_2')$ would yield a different value than that calculated from $p(a_1, a_5)$ or $p(a_1, a_2)$ (similarly for A_5'). The consequence of this would be that causality is violated, as can be seen from the following argument. In order to evaluate probabilities, joint measurements do not have to be performed simultaneously, they can as well be performed in sequential order. The fact that $p(a_1)$ depends on whether A_1 was measured with A_1', A_2' or with A_2 or A_5 can be used to signal backward in time. The marginal probabilities $p(a_1)$ calculated from $p(a_1, a_1', a_2')$ being inconsistent with the probability $p(a_1)$ measured earlier (in a joint measurement of A_1 and A_2, for instance) would imply an influence propagating backward in time, thus violating causality [15]. The no-signaling principle being a special instance of the no-disturbance principle, violation of no-signaling monogamies for Bell inequalities would imply the possibility of superluminal communication between spatially separated systems, which could also lead to a violation of causality. In the Supplemental Material [16], we help clarify the analogous relationship between the monogamy of contextual inequalities derived here and the monogamy of Bell inequalities (which are special instances of contextual inequalities) by deriving the monogamy of Clauser-Horne-Shimony-Holt inequalities using the commutation graph technique introduced above.

Having illustrated the method for deriving monogamy relations for contextual (and Bell) inequalities, we now proceed to formulate it using some graph-theoretic notions. To do so, we first state Proposition 1, whose proof is provided in the Supplemental Material [16].

Proposition 1.—A commutation graph G representing a set of n measurements (for any n) admits a joint probability distribution for these measurements if it is a chordal graph.

A chordal graph is a graph that does not contain an induced cycle of length greater than 3; i.e., each of the cycles of four or more vertices in the graph must have a chord, an edge connecting two nonadjacent vertices in the cycle. This class of graphs comprises a large class of all graphs of n vertices, and Proposition 1 excludes the construction of contextual inequalities (or Kochen-Specker proofs) from all such graphs.

We can now precisely formulate the method for the derivation of monogamy relations introduced previously. Given a commutation graph representing a set of n contextual inequalities, look for its vertex decomposition into m chordal subgraphs (each of which admits a joint

probability distribution by Proposition 1), such that the sum of the independence numbers of these subgraphs is $n * R$, where R is the noncontextual bound for each of the inequalities. If the n contextual inequalities are not all the same, i.e., if n_1 inequalities have noncontextual bound R_1 and n_2 inequalities have bound R_2 etc., then the subgraphs should be chosen such that the sum of their independence numbers is $\sum_k n_k R_k$. All vertices of the commutation graph are to be included in the vertex decomposition into subgraphs with no vertex appearing in more than one subgraph, but the edges between the different subgraphs can be neglected. Note that while many contextual inequalities involve rank-1 projectors, where the edges of the graph denote mutual exclusiveness in addition to compatibility, this assumption is not crucial to the derivation of monogamies. This can be seen from the derivation of the Bell inequality monogamies, where only compatibility is required.

Using the method presented above, one can identify several commutation graphs that yield contextual monogamy (including Bell monogamy) relations; for instance, the monogamy relation Eq. (3) also holds for the graphs in Figs. 2(a) and 2(b), as can be seen by the decompositions given there. We see that monogamy relations for two KCBS inequalities can be derived for various measurement configurations, the measurement configuration given in Fig. 1 being the minimal one (with fewest edges connecting two contextual graphs) in which such monogamies appear for two sets of five separate measurements. This minimality can be seen by finding that for all graphs with one, two, and three edges connecting two distinct pentagon graphs, no vertex decomposition into two or more chordal subgraphs with total independence number 4 exists. Since the KCBS inequality Eq. (2) is a necessary and sufficient condition for the existence of noncontextual description for the five measurements, the relation Eq. (3) holds for any contextual inequality of this kind as well [14]. The method can also be used to construct monogamy relations for inequalities with more than five measurements along similar lines. In general, it can be seen that the larger the number of mutually exclusive and jointly performable measurements, the stronger the monogamy relation is.
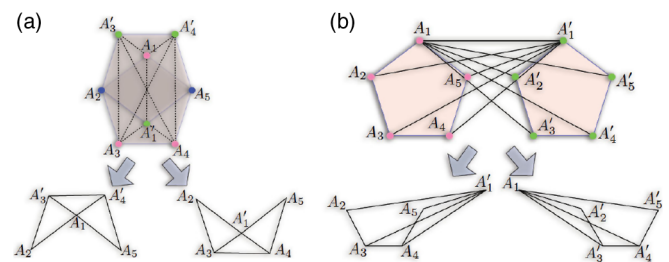

FIG. 2 (color online). Measurement configurations (top) and their decompositions (bottom) for which monogamy relations can be derived.

We now proceed to explicitly identify the commutation graphs that give rise to monogamy relations for a given set of n KCBS-type contextual inequalities (with noncontextual bound R). This is done in Proposition 2, which provides the necessary and sufficient condition for a commutation graph to give rise to a monogamy relation using the method outlined above; its proof is provided in the Supplemental Material [16].

Proposition 2.—Consider a commutation graph representing a set of n KCBS-type contextual inequalities, each of which has noncontextual bound R. Then this graph gives rise to a monogamy relation using the outlined method if and only if its vertex clique cover number is $n * R$.

The vertex clique cover number is the minimal number of cliques required to cover all the vertices of the graph. The above proposition can be extended to the scenario where one is interested in the monogamy of sets of different contextual inequalities, for instance, when n_k of the inequalities have noncontextual bound R_k, with $\sum_k n_k = n$. Then the condition becomes that the vertex clique cover number equals $\sum_k n_k R_k$. This gives a very powerful method of identifying whether a given graph exhibits contextual monogamy.

The monogamy relations presented so far are genuine properties of contextual theories in the sense that, for classical (noncontextual) theories, each of the inequalities can achieve its maximum value within the classical theory. We note, however, that certain monogamies also hold for noncontextual theories; for instance, for the situation when all measurements in set $\{A_i\}$ are compatible and mutually exclusive with all the measurements in set $\{A_i'\}$. Here, the mutual exclusiveness guarantees the monogamy $\sum_{i=1}^5 p(A_i = 1) + \sum_{i=1}^5 p(A_i' = 1) \leq 5/2$ in all theories obeying the no-disturbance principle. However, an important feature here is that monogamies also arise within noncontextual theories for which the relation $\sum_{i=1}^5 p(A_i = 1) + \sum_{i=1}^5 p(A_i' = 1) \leq 2$ holds, and this can be traced to the large number of mutually exclusive measurements required here. The interesting monogamies are those in which such classical restrictions do not appear, such as those in Figs. 1 and 2.

Let us now show how the monogamy relation Eq. (3) applies within quantum theory. First, note that the measurements for the optimal violation of KCBS inequality for a single three-level quantum system are rank-1 projectors spanning real three-dimensional space. Consider a real four-dimensional space in which the set of projectors $\{A_i\}$ spans dimensions 1, 2, and 3 and the set of projectors $\{A_i'\}$ spans dimensions 2, 3, and 4. These projectors can be constructed to obey the constraints of mutual exclusiveness and joint measurability as required by the commutation graphs. A set of projectors that correspond to the measurement configuration in Fig. 2(b) for a quantum mechanical system of dimension four is given by

$$|A_1\rangle = (1, 0, 0, 0)^T, \qquad |A_2\rangle = (0, 1, 0, 0)^T,$$

$$|A_3\rangle = (\cos\theta, 0, \sin\theta, 0)^T,$$

$$|A_4\rangle = (\sin\alpha \sin\theta, \cos\alpha, -\sin\alpha \cos\theta, 0)^T,$$

$$|A_5\rangle = \frac{1}{\sqrt{\cos^2\alpha + \sin^2\alpha\cos^2\theta}}(0, \sin\alpha \cos\theta, \cos\alpha, 0)^T,$$

and

$$|A_1'\rangle = (0, 0, 0, 1)^T, \qquad |A_2'\rangle = (0, \cos\beta, \sin\beta, 0)^T,$$

$$|A_3'\rangle = (0, \sin\gamma \sin\beta, -\sin\gamma \cos\beta, \cos\gamma)^T,$$

$$|A_4'\rangle = (0, \sin\delta \sin\varepsilon, -\sin\delta \cos\varepsilon, \cos\delta)^T,$$

$$|A_5'\rangle = \frac{1}{\sin\delta}(0, \sin\delta \cos\varepsilon, \sin\delta \sin\varepsilon, 0)^T,$$

where we impose the conditions $\sin(\beta - \varepsilon) \neq 0$, $\cos(\beta - \varepsilon) \neq 0$, and $\tan\delta \tan\gamma \cos(\beta - \varepsilon) = -1$. This is analogous to the situation where one requires three qubits (dimension 8), which is the minimal system in order to have monogamy of Bell inequality violations. Since quantum theory obeys the principle of no disturbance, the monogamy inequality Eq. (3) is guaranteed to hold for these projectors. However, quantum mechanics incorporates other properties as well, such as the complementarity principle [17]. For the monogamy relations of Bell inequalities, the exact trade-offs between multiple inequalities within quantum theory have been derived [17,18] using the principle of complementarity. An important open problem is to derive the exact trade-offs for contextual inequalities within quantum theory.

Discussion and conclusions.—Violation of a single contextual inequality Eq. (2) for a three-level system has been experimentally tested using a single photon [8]. The monogamy of contextuality presented here can be realized for a four-level system with feasible modifications to the existing experimental setup, and using projectors according to the commutation graphs presented in the figures above. This would establish the monogamy of contextuality as a distinct phenomenon from the monogamy of entanglement, since the notion of entanglement is not clearly applicable to a single quantum system.

In this Letter we have demonstrated that the property of monogamy so far seen for quantum correlations in composite entangled systems also carries over to single quantum systems. In particular, we have shown that one can construct monogamy relations for contextual inequalities of the KCBS type using the principle of no disturbance. Violation of these monogamy relations is seen to be related to a violation of causality analogous to the necessity of signaling between spatially separated systems for the violation of Bell monogamy relations. The fact that Bell inequalities and contextual inequalities arise from the same origin, namely, the assumption of realism, leads one to believe that the features seen in the Bell scenario should carry over to the contextual scenario as well. In this regard, it would be interesting to investigate

how other features such as distillation, activation of non-locality, and no-signaling boxes (or Popescu-Rohrlich boxes) carry over to the contextual scenario [13].

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*phykd@nus.edu.sg

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