

Solutions of the Vector Nonlinear Schrödinger Equations: Evidence for Deterministic Rogue Waves

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We construct and discuss a semirational, multiparametric vector solution of coupled nonlinear Schrödinger equations (Manakov system). This family of solutions includes known vector Peregrine solutions, bright- and dark-rogue solutions, and novel vector unusual freak waves. The vector rogue waves could be of great interest in a variety of complex systems, from optics and fluid dynamics to Bose-Einstein condensates and finance.

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Introduction.—Extreme wave events, also referred to as freak or rogue waves (terms which are meant here as synonymous), are mostly known as oceanic phenomena responsible for a large number of maritime disasters. These waves, which have height and steepness much greater than expected from the sea average state [1], have recently become a topic of intense research. Freak waves appear both in deep ocean and in shallow water [2]. In contrast to tsunamis and storms associated with typhoons that can be predicted hours (sometimes days) in advance, the particular danger of oceanic rogue waves is that they suddenly appear from nowhere only seconds before they hit a ship. The grim reality, however, is that although the existence of freak waves has now been confirmed by multiple observations, uncertainty remains on their fundamental origins. This hinders systematic approaches to study their characteristics, including the predictability of their appearance [3].

In fact, research on rogue waves is in an emerging state [1,3,4]. These waves not only appear in oceans [5] but also in the atmosphere [6], in optics [7,8], in plasmas [9], in superfluids, in Bose-Einstein condensates [10], and also as capillary waves [11]. The common features and differences among freak wave manifestations in their different contexts is a subject of intense discussion [2]. New studies of rogue waves in any of these disciplines enrich their concept and lead to progress towards a comprehensive understanding of a phenomenon which still remains largely unexplored. A formal mathematical description of a rogue wave is provided by the so-called Peregrine soliton [12]. This solitary wave is a solution of the 1 + 1 scalar nonlinear Schrödinger equation (NLSE) with the property of being localized in both coordinates: thus it describes a unique wave event. This solution is also unique in a mathematical sense, as it is written in terms of rational functions of coordinates, in contrast to most of other known solutions of the NLSE which are instead purely exponential. Recent experiments have provided a path to generating Peregrine solitons in optical fibers with standard telecommunication equipment [13]. The further experimental observation of

Peregrine solitons in a water tank [14] indicates that they can also describe rogue waves in oceans. The Peregrine soliton is not the only fully localized waveform [15]. In fact, there is an infinite hierarchy of rational solutions of the NLSE which enjoy the same property [16–19].

In a variety of complex systems such as Bose-Einstein condensates [20], optical fibers [21], and financial systems [22,23], several amplitudes rather than a single one need to be considered. The resulting systems of coupled equations may thus describe extreme waves with higher accuracy than the scalar NLSE model. Approaches to rogue wave phenomena involving more than one wave amplitude are the vector Gross-Pitaevskii (GP) model [20,24] and the Manakov system [25]. Indeed, vector rogue wave solutions of GP equations and of the Manakov system have been recently presented [20,23,26].

In this Letter, we contribute to the field of rogue waves by constructing a new multiparametric vector soliton solution of the Manakov system. A key novel property of this solution is that it features both exponential and rational dependence on coordinates and, therefore, it is called semirational. For special parameter values, our general solution reproduces known rogue waves, such as the vector Peregrine soliton and bright- and dark-rogue waves [26]. Our treatment below goes as follows. We give the essential Darboux dressing transformation to construct freak solutions of the Manakov system. We present the expression of multiparametric, semirational deterministic freak waves. Moreover, we discuss their experimental feasibility in nonlinear optics.

Darboux dressing technique.—Waves are assumed to be modeled by the dimensionless vector nonlinear Schrödinger equations (VNLSE) or Manakov system:

$$\begin{aligned} iu_t^{(1)} + u_{xx}^{(1)} + 2(|u^{(1)}|^2 + |u^{(2)}|^2)u^{(1)} &= 0 \\ iu_t^{(2)} + u_{xx}^{(2)} + 2(|u^{(1)}|^2 + |u^{(2)}|^2)u^{(2)} &= 0, \end{aligned} \quad (1)$$

where each subscripted variable in Eq. (1) stands for partial differentiation. Moreover $u^{(1)}(x, t)$, $u^{(2)}(x, t)$ represent the

wave envelopes, t is the evolution variable, and x is a second independent variable. It should be pointed out that the meaning of the dependent variables $u^{(1)}(x, t)$, $u^{(2)}(x, t)$, and of the coordinates t , x depends on the particular applicative context (e.g., fluid dynamics, plasma physics, nonlinear optics, finance). Note also that Eqs. (1) refer to the self-focusing (or anomalous dispersion) regime. Eqs. (1) are integrable: the associated Lax pair is

$$\begin{aligned}\Psi_x &= (ik\sigma + Q)\Psi, \\ \Psi_t &= [2ik^2\sigma + 2kQ + i\sigma(Q^2 - Q_x)]\Psi,\end{aligned}\quad (2)$$

where $\Psi = \Psi(x, t, k)$ is a 3×3 matrix solution, k is the complex spectral variable, the matrix $\sigma = \text{diag}\{1, -1, -1\}$ is constant and diagonal, and $Q = Q(x, t)$ is the 3×3 matrix

$$Q = \begin{pmatrix} 0 & -u^{(1)*} & -u^{(2)*} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix}. \quad (3)$$

The starting point here is the construction of the solution representing one soliton wave nonlinearly superimposed to the following plane wave background solution of Eqs. (1)

$$\begin{pmatrix} u_0^{(1)}(x, t) \\ u_0^{(2)}(x, t) \end{pmatrix} = e^{2i\omega t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad (4)$$

where a_1 and a_2 are arbitrary parameters which, with no loss of generality, are taken real. Moreover the frequency ω reads as $\omega = a^2$ where, from now on, we set $a = \sqrt{a_1^2 + a_2^2}$. The Darboux method to obtain the one-soliton solution $u^{(1)}(x, t)$, $u^{(2)}(x, t)$ is well known, therefore we omit detailed computations, limiting ourselves to list the essential few steps. Our results have been obtained by following the general formulation and construction as presented in [27] (the interested reader may find additional references quoted there).

The chain of calculations ends up with the following general formula [27]

$$\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = e^{2i\omega t} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{2i(\beta^* - \beta)\zeta^*}{|\zeta|^2 + z^\dagger z} \begin{pmatrix} z^{(1)} \\ z^{(2)} \end{pmatrix}. \quad (5)$$

The constant parameter β is complex (with nonvanishing imaginary part), while the functions $\zeta(x, t)$, $z^{(1)}(x, t)$, $z^{(2)}(x, t)$ are the components of a generic three-dimensional vector solution $Z(x, t)$ of the Lax pair of equations (2), which corresponds to the spectral parameter $k = \beta$ and to the background solution (4). Thus, if Z_0 is a constant arbitrary complex three-dimensional vector, $Z(x, t)$ reads as

$$Z = \begin{pmatrix} \zeta \\ z^{(1)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\omega t} & 0 \\ 0 & 0 & e^{2i\omega t} \end{pmatrix} \exp(i\Lambda x - i\Omega t) Z_0, \quad (6)$$

where Λ and Ω are the constant matrices

$$\begin{aligned}\Lambda &= \begin{pmatrix} \beta & ia_1 & ia_2 \\ -ia_1 & -\beta & 0 \\ -ia_2 & 0 & -\beta \end{pmatrix}, \\ \Omega &= -\Lambda^2 - 2\beta\Lambda + \beta^2 + 2a^2.\end{aligned}\quad (7)$$

Equation (6) shows that, if the matrix Λ (and therefore Ω) possesses three linearly independent eigenvectors, then the vector $Z(x, t)$ is a linear combination of exponential functions of x and t . Therefore, the solution (5) cannot be rational or semirational. On the contrary, if Λ and Ω cannot be reduced to diagonal form but are similar to a Jordan form, then the exponential $\exp(i\Lambda x - i\Omega t)$ [see (6)] becomes a combination of exponential and polynomial functions of x and t . Therefore, our task is that of finding those particular values of β [see (7)] such that the matrices Λ and Ω are similar to a Jordan matrix. Indeed this happens if and only if $\beta = \pm ia$. By taking, e.g., $\beta = ia$, in this way we arrive at the following semirational solution of the VNLSE Eqs. (1)

$$\begin{pmatrix} u^{(1)}(x, t) \\ u^{(2)}(x, t) \end{pmatrix} = e^{2i\omega t} \left[\frac{L}{B} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{M}{B} \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} \right], \quad (8)$$

with the following notation: $L = \frac{3}{2} - 8\omega^2 t^2 - 2a^2 x^2 + 8i\omega t + |f|^2 e^{2ax}$, $M = 4f(ax - 2i\omega t - \frac{1}{2})e^{(ax+i\omega t)}$, and $B = \frac{1}{2} + 8\omega^2 t^2 + 2a^2 x^2 + |f|^2 e^{2ax}$, where f is a complex arbitrary constant. It should be remarked that the dressing construction of the vector rogue wave (8) has introduced as arbitrary parameters the three complex components of the vector Z_0 ; see (5) and (6). However only the complex parameter f is left essential out of these components, since the other parameters enter as trivial translations of the coordinates x , t , which have been set to zero for simplicity. The two other real parameters a_1 , a_2 originate instead from the naked solution, namely from the background plane wave (4). We note also that the dependence of L , M and B [see Eq. (8)] on x , t is both polynomial and exponential only through the dimensionless variables ax and $\omega t = a^2 t$. Moreover the vector solution (8) turns out to be a combination of the two constant orthogonal vectors $(a_1, a_2)^T$ and $(a_2, -a_1)^T$.

Vector semirational rogue waves.—Setting $f = 0$ implies $M = 0$: in this particular case the expression (8) yields the trivial vector generalization of the rational Peregrine solution [12,20]. In this case $u^{(1)}(x, t)$ is merely proportional to $u^{(2)}(x, t)$. We note that the amplitude $|u^{(j)}(x, t)|$ is peaked at $x = 0$ with the maximum value $3|a_j|$ at $t = 0$ (see Fig. 1).

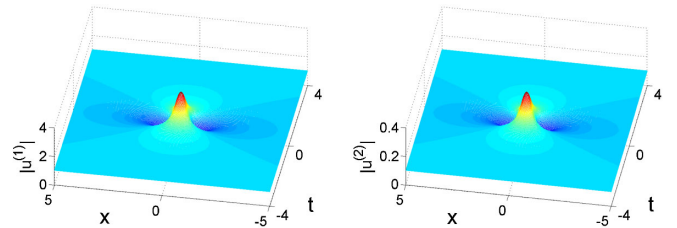


FIG. 1 (color online). Deterministic vector freak waves envelope distributions $|u^{(1)}(x, t)|$ and $|u^{(2)}(x, t)|$ of (8). Here, $a_1 = 1$, $a_2 = 0.1$, $f = 0$.

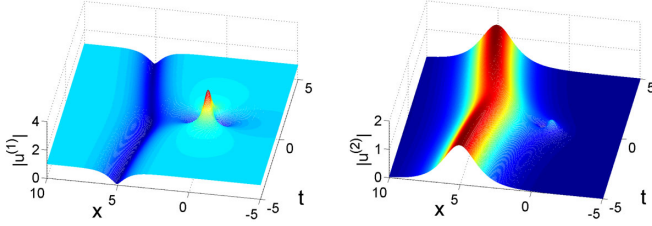


FIG. 2 (color online). As in Fig. 1, with $f = 0.1$, $a_1 = 1$, $a_2 = 0$.

If instead $f \neq 0$, the Peregrine bump coexists and interacts with a pulse which propagates with nonconstant speed, and, depending on the value of $|f|$, may have different looks. In order to better describe this behavior, we note that the ratios $L(x, t)/B(x, t)$ and $M(x, t)/B(x, t)$ which appear in (8) describe asymptotically as $t \rightarrow \pm\infty$ a dark and a bright pulse, respectively. Thus each wave component $u^{(j)}(x, t)$ is generically a mixture of a dark and a bright pulse. Leaving aside the detailed analytic description of the solution (8) at intermediate times, we limit our present analysis to the large time behavior. The pulse motion, for each individual dark and bright contribution, asymptotically reads as

$$x = \xi(t) \rightarrow x_0 + \frac{1}{a} \ln(\omega|t|) + O\left(\frac{\ln^2(|t|)}{t^2}\right), \quad t \rightarrow \pm\infty, \quad (9)$$

where $x_0 = (1/a) \ln(2\sqrt{2}/|f|)$. This implies that this pulse goes to infinity where it “stops” since its velocity slowly vanishes as $d\xi/dt \rightarrow 1/(at)$. The shape of the dark and bright contributions at large times as a function of the parameter χ which measures the displacement from the peak position, takes the expected form as $t \rightarrow \pm\infty$

$$\frac{L}{B} \rightarrow \tanh(\chi), \quad \frac{M}{B} \rightarrow -i\sqrt{2} \left(\frac{ft}{|ft|} \right) \frac{e^{i\omega t}}{\cosh(\chi)}. \quad (10)$$

The superposition of the dark and bright contributions in each of the two wave components $|u^{(j)}|$ may give rise to complicated breather-like pulses. These results are well represented in Figs. 2–4. The single contributions of the dark shape L/B and bright shape M/B are better displayed when, e.g., $a_2 = 0$. In this case, typical distributions $|u^{(1)}(x, t)|$, $|u^{(2)}(x, t)|$ are displayed in Figs. 2 and 3. Figure 2 shows a vector dark-bright soliton together with a single Peregrine soliton. Decreasing the value of $|f|$, Peregrine and dark-bright solitons separate. By increasing $|f|$, Peregrine and dark-bright solitons merge and the Peregrine bump cannot be identified while the resulting dark-bright pulse

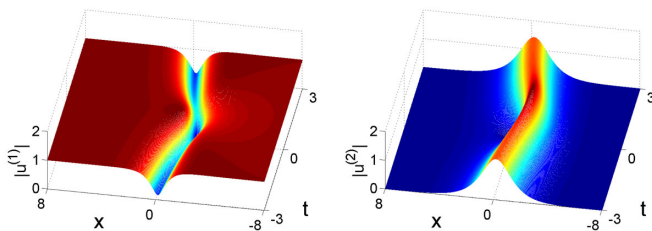


FIG. 3 (color online). As in Fig. 1, with $f = 10$, $a_1 = 1$, $a_2 = 0$.

appears as a boomeron-type soliton (see Fig. 3), i.e., a soliton solution with a time-dependent velocity [28,29]. Note that the solution (8) includes, as a special case, the bright-dark-rogue wave solution that was reported in [26].

Finally, our formula (8), if all parameters f , a_1 , a_2 are nonvanishing, describes the dynamics of a breather-like wave resulting from the interference between the dark and bright contributions. Distributions $|u^{(1)}(x, t)|$, $|u^{(2)}(x, t)|$ which are typical of this general case are displayed in Fig. 4. Again, by decreasing $|f|$, Peregrine and breather solitons separate, while if $|f|$ increases, Peregrine and breather solitons merge, with boomeronic behavior.

These results provide evidence of an attractive interaction between the dark-bright wave and the Peregrine rogue wave. The observed behavior may also be interpreted as a mechanism of generation of one rogue wave out of a slowly moving boomeronic soliton.

Let us discuss the experimental conditions for the observation of the vector, semirational freak solitons. Nonlinear optics is a fertile ground to develop the knowledge of the phenomenon of vector freak or rogue waves. As first scenery, consider the propagation of arbitrarily polarized optical pulses in a weakly dispersive and nonlinear dielectric. In fact, Eq. (1) applies to a Kerr medium with the electrostrictive mechanism of nonlinearity [30], as well as to randomly birefringent fiber optic transmission links [31,32]. Indeed, the use of the polarization degree of freedom for doubling the capacity of long-distance, fiber-based transmission systems has been widely adopted by means of the technique of polarization multiplexing. To be specific, we consider the transmission at the 40 Gbit/s rate of a train of dark-bright solitons, dark in one polarization, and bright in the orthogonal polarization. Figure 5 shows that a Peregrine soliton is generated at 800 km (here the evolution variable t is the propagation distance and the variable x is the time), and it attracts a dark-bright soliton. In this example, we numerically integrated Eq. (1) for properly rescaled wave envelope amplitudes E_Y , E_X , and rescaled coordinates, with initial conditions two dark-bright solitons plus a small noise seed. We used a fiber nonlinear coefficient of $1.3 \text{ km}^{-1} \text{ W}^{-1}$, the anomalous average fiber dispersion of $0.1 \text{ ps}^2/\text{km}$, and a dark-bright full width at half maximum of 8.25 ps (33% of the 25 ps bit period); the peak power of the two polarizations is equal to 3 and 6 mW, respectively. As second scenery, we may consider incoherently coupled photorefractive spatial waves in strontium barium niobate (SBN). Modulation

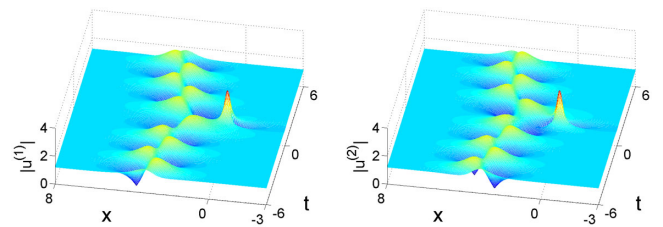


FIG. 4 (color online). As in Fig. 1, with $f = 0.1i$, $a_1 = 1.2$, $a_2 = 1.2$.

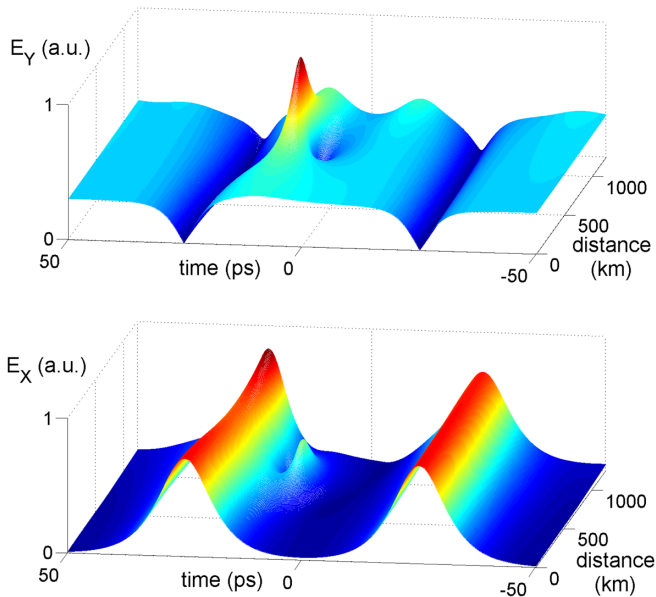


FIG. 5 (color online). Numerical transmission of two 50 ps spaced dark-bright solitons in optical fibers, y -polarized dark waves (E_Y), and x -polarized bright envelopes (E_X).

instability and the existence of unstable dark-bright pairs (first steps in demonstrating vector Peregrine waves and dark and bright Peregrine dynamics) have been already demonstrated in SBN [33]. Setups proposed in Refs. [33,34] can be exploited to observe and characterize spatial vector rogue waves in SBN.

Conclusions.—Here, we have analytically constructed and discussed a novel multiparametric vector freak solution of the VNLSE. This family of exact solutions includes known vector Peregrine (rational) solutions, as well as new freak wave solutions which feature both exponential and rational dependence on coordinates. Because of the universality of the VNLSE (1), which models the coupling of two nonlinear waves, our solutions contribute to better control and understanding of rogue wave phenomena in a variety of complex dynamics, ranging from fluid dynamics, to optical communications, Bose-Einstein condensates, and financial systems.

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