Quantized Adiabatic Transport In Momentum Space

Derek Y. H. Ho[1] and Jiangbin Gong[1,2,*]

[1]*Department of Physics and Center for Computational Science and Engineering, National University of Singapore, 117542, Singapore*
[2]*NUS Graduate School for Integrative Sciences and Engineering, Singapore 117597, Singapore*
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Though topological aspects of energy bands are known to play a key role in quantum transport in solid-state systems, the implications of Floquet band topology for transport in momentum space (i.e., acceleration) have not been explored so far. Using a ratchet accelerator model inspired by existing cold-atom experiments, here we characterize a class of extended Floquet bands of one-dimensional driven quantum systems by Chern numbers, reveal topological phase transitions therein, and theoretically predict the quantization of adiabatic transport in momentum space. Numerical results confirm our theory and indicate the feasibility of experimental studies.

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In both classical mechanics and quantum mechanics, position and momentum variables form a conjugate pair and can hence be treated on the same footing from a phase space perspective. The real physical world, however, does not have position-momentum symmetry. For example, energy bands of a solid are formed because electronic Hamiltonians are periodic in position but not in momentum. Because of such unequal roles of position and momentum, the mapping of quantum transport phenomena from position space to momentum space is typically nontrivial but, where possible, may lead to important insights and unforeseen opportunities. Anderson localization, for instance, was first discovered as a seminal result of quantum transport in position space. Its analog in momentum space was later found to be behind the intriguing phenomenon of "dynamical localization" [1,2]. This mapping has stimulated fruitful studies of Anderson transition in driven cold-atom systems [3]. As a second example, ratchet transport, namely, directed transport under a zero mean force in position space, has been mapped to momentum space as well, leading to the finding of ratchet accelerators (RA) [4–6].

Energy-band topology is of fundamental interest to studies of quantum transport in condensed-matter physics. The issue to be addressed here is whether transport in momentum space (i.e., acceleration) can be connected with band structures due to momentum-space periodicity. At first glance this sounds impossible because, with the (nonrelativistic) kinetic energy being a quadratic function of momentum, a realistic Hamiltonian is never a periodic function of momentum. However, for systems periodically driven by impulsive fields, the Floquet operator can still be periodic in momentum, thus forming Floquet (quasienergy) bands. The topology of such Floquet bands then provides a useful tool in characterizing topological phase transitions in driven quantum systems [7]. How Floquet-band topology is manifested in acceleration then becomes an intriguing question.

Using a one-dimensional RA model inspired by existing cold-atom experiments, we show in this work that the Floquet bands, defined on a 2-torus (formed by one Bloch phase and one experimentally tunable parameter) may be characterized by Chern numbers. We then theoretically show that adiabatic transport in *momentum space* can be quantized according to these topological numbers. Unlike quantized adiabatic pumping in position space [8], there does not exist a general (system-independent) flux operator in momentum space and hence the found quantization is about a quantized net change of momentum expectation value, rather than a pumping of particles through a cross section. Finally, though adiabatic manipulation of Floquet states is often considered to be subtle [9,10], our numerical results confirm our theory and indicate that quantized momentum-space transport can be observed in a wide regime, by simply scanning one system parameter in a relatively small number of discretized steps. Breakdown of quantization in momentum-space transport may then be considered as a diagnostic tool for detecting nonadiabatic effects in Floquet state manipulation.

To physically realize the main physics we need three ingredients: momentum-space periodicity, an experimentally tunable periodic parameter and well-gapped Floquet bands. To be specific we consider a RA model [11], which is based on an atom-optics realization of a double-kicked rotor system [12]. The RA Hamiltonian is given by $H=\frac{p^2}{2}+K\cos(q+\alpha)\sum_n\delta(t-nT)+K\cos(q)\times\sum_n\delta(t-nT-T_0)$, with a corresponding Floquet propagator [11] $\hat{U}(\alpha)=e^{-i(T-T_0)(p^2/2\hbar)}e^{-i(K/\hbar)\cos(q)}e^{-iT_0(p^2/2\hbar)}\times e^{-i(K/\hbar)\cos(q+\alpha)}$, where all quantities are properly scaled and hence in dimensionless units, q and p are canonical coordinate and momentum operators for cold atoms, and the δ kicks are of period T, experimentally implemented by two optical-lattice potentials mutually phase-shifted by α, with equal strength K, and a time lag $T_0 < T$ [12]. Note that later α will be adiabatically tuned. Because the potential function is periodic in q, momentum eigenstates take eigenvalues $m\hbar + \beta\hbar$, where $\beta \in [0, 1]$ is a conserved quasimomentum variable and m is an integer. To yield a Floquet operator periodic in momentum despite the $p^2/2$ term in the Hamiltonian, we set $\beta = 0$, which may be approximately implemented by considering a Bose-Einstein condensate whose coherence width spans across many optical-lattice constants [6,13,14]. Effects of nonzero β values will be discussed in Supplemental Material [15]. If we now impose the quantum resonance condition $T\hbar = 4\pi$ that has been one experimental subject [6,13,14,16–18], we obtain an "on-resonance double-kicked-rotor model" [20], with the Floquet propagator reducing to

$$\hat{U}_R(\alpha) = e^{i(p_e^2/2\hbar_e)} e^{-i(K_e/\hbar_e)\cos(q)} e^{-i(p_e^2/2\hbar_e)} e^{-i(K_e/\hbar_e)\cos(q+\alpha)}, \tag{1}$$

where $\hbar_e \equiv \hbar T_0$ is an effective Planck constant, $K_e \equiv KT_0$, and a rescaled momentum operator $p_e \equiv T_0 p$. From here on, momentum exclusively refers to p_e and we denote momentum eigenstates by $|m\rangle$, which has eigenvalue $m\hbar_e$ and is periodic in q with period 2π. Equation (1) indicates that if $\hbar_e = 2\pi M/N$, with M and N being integers, then the Floquet operator $\hat{U}_R$ is perfectly periodic in momentum space with a period of $N\hbar_e$. According to Bloch's theorem, this momentum-space periodicity leads to Floquet bands. Indeed, in the case of $\alpha = 0$, the corresponding Floquet band structure [19] resembles Hofstadter's butterfly [20], with $\hbar_e$ here identified as an analog of the magnetic flux. It is such a remarkable resemblance (which also hints desirable band-gap features) that motivated us to connect the topological aspects of the Floquet bands with momentum-space transport.

The Floquet band structure may be characterized by topological Chern numbers, provided that the bands are defined on a 2-torus. To that end, we supplement the Bloch phase parameter in momentum space with the periodic parameter α. This procedure somewhat resonates with recent efforts in identifying analogs of quantum Hall effect in one-dimensional systems [21]. The eigenstate-eigenvalue problem for $\hat{U}_R$ now becomes $\hat{U}_R(\alpha)|\psi_n(\phi, \alpha)\rangle = e^{i\omega_n(\phi,\alpha)}|\psi_n(\phi, \alpha)\rangle$, where $\phi \in [0, 2\pi)$ is the Bloch phase in momentum space and $\omega_n(\phi, \alpha)$ is the eigenphase of $\hat{U}_R(\alpha)$. For a fixed pair of α and ϕ, N eigenphases can be obtained, and scanning (ϕ, α) over $[0, 2\pi) \times [0, 2\pi)$ forms *extended* Floquet bands on 2-torus, so-named to distinguish them from the common bands involving only the Bloch phase parameter ϕ. As a result n ($1 \le n \le N$) becomes an index of such bands (with $\hbar_e = 2\pi M/N$). The eigenstates $|\psi_n(\phi, \alpha)\rangle$ are chosen such that they are locally single-valued functions of α and ϕ, and periodic functions of ϕ. Two computational examples of the Floquet bands $\omega_n(\phi, \alpha)$ are depicted in Fig. 1 for $N = 3$. It is seen that as the driving strength K varies, the landscape of the bands changes. Other calculations show that if $\hbar_e = 2\pi M/N$ with even

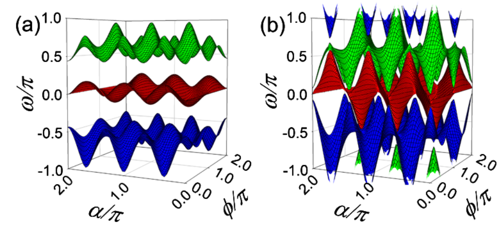

FIG. 1 (color online). Floquet eigenphase $\omega_n(\phi, \alpha)$ vs ϕ and α for (a) $K_e = 3\hbar_e$ and (b) $K_e = 4\hbar_e$.

N, then there will be two (extended) bands touching each other. For simplicity, here we consider only odd N so that only accidental band collisions occur.

In Fig. 2 we present the Chern numbers (see its surface integral expression below and see [15] for computational details) for a 3-band case. It is seen that at some isolated critical values of K_e, the Chern numbers jump, signaling the presence of toplogical phase transitions in our RA model. Indeed, the Chern numbers are invariant integers with respect to smooth deformation of the bands and only change discontinuously due to band collisions. Note that mainly in the context of quantum-classical correspondence in classically chaotic systems, Ref. [22] (see also Ref. [23]) formally studied the topological aspects of Floquet bands defined on a 2-torus formed by Bloch phases in both position space and momentum space. Our study is much different because (i) this work is based on an explicit physical implementation of momentum-space periodicity, (ii) here the Bloch phase in position space is fixed at $\beta = 0$ in theory (so as to obtain Hofstadter's butterfly Floquet spectrum), and (iii) our Floquet bands are defined on a 2-torus that involves one experimental parameter, a key starting point for our theory below.

We now seek the implications of the Floquet-band Chern numbers by considering an adiabatic cycle during which α increases from 0 to 2π. We first construct an initial state of the following form,

$$|\Psi_n(\alpha = 0)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi |\psi_n(\phi, \alpha = 0)\rangle, \tag{2}$$

which represents an equal-weight superposition of all the Floquet eigenstates of band n with $\alpha = 0$. This coherent superposition state, which can be interpreted as a Wannier function in momentum space, uniformly samples all the

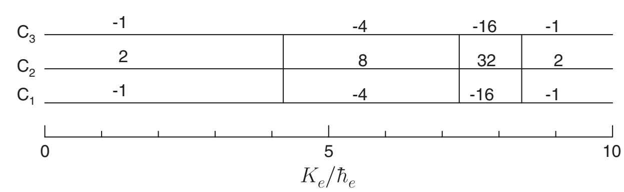

FIG. 2. The Chern numbers of the 3 bands vs $K_e/\hbar_e$, for $\hbar_e = 2\pi/3$. For results of a 7-band case, see the Supplemental Material [15].

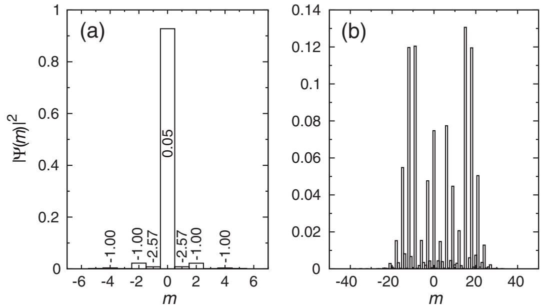

FIG. 3. Momentum distribution $|\langle m|\Psi_3(\alpha)\rangle|^2 \equiv |\Psi(m)|^2$ [see Eq. (2)] with $K_e = 2\hbar_e$ for $t = 0$ in (a) and after a 100-period adiabatic cycle in (b). Numbers shown in (a) are the phases of each momentum component.

Bloch eigenstates with different values of ϕ (but all with $\alpha = 0$), with a profile localized in the momentum space. Indeed, each eigenstate $|\psi_n(\phi, \alpha = 0)\rangle$ is infinitely extended in momentum space, but $|\Psi_n(\alpha = 0)\rangle$ is normalized to unity (localized in momentum space with Gaussian-like tails in all the cases we studied). It is worth noting that because each eigenstate $|\psi_n(\phi, \alpha = 0)\rangle$ is defined only up to a global phase, one is free to choose an overall phase convention of $|\psi_n(\phi, \alpha)\rangle$ such that the superposition state in momentum space tends to be well-localized, thus making experimental preparation of the initial state easier. Such states can be highly localized so long as K is not too large. For example, the shown superposition state in Fig. 3(a) mainly occupies one momentum eigenstate, with small weights distributed over only a few nearby components. Given previous experiments where momentum superposition states in the same context were prepared [6], states as shown in Fig. 3(a) should be reachable in experiments.

Consider then an adiabatic change in α, through a discretized protocol $\alpha_s = 2\pi s/s_f$ for $(s-1)T \le t < sT$, so that α completes one adiabatic cycle at $t = s_f T$ [24]. Assuming adiabatic following of the Floquet states, the state evolved from $|\Psi_n(\alpha = 0)\rangle$ [see Eq. (2)] should stay as a superposition state at $t = (sT)^-$, with each component still being the eigenstate of $\hat{U}_R(\alpha)$, with $\alpha = \alpha_s$. That is, at $t = (sT)^-$, under adiabatic approximation the associated time-evolving state of the system becomes

$$|\Psi_n(\alpha_s)\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi |\psi_n(\phi, \alpha_s)\rangle e^{i\theta(\phi,\alpha_s)}, \qquad (3)$$

where $\theta(\phi, \alpha_s)$ is the sum of a dynamical phase and a geometrical phase accumulated by the component starting from $|\psi_n(\phi, \alpha = 0)\rangle$. As a consequence of choosing $|\psi_n(\phi, \alpha = 0)\rangle$ to be a periodic function of ϕ, $\theta(\phi, \alpha_s)$ is also necessarily periodic in ϕ.

Next we evaluate $\langle p_e(s)\rangle \equiv \langle\Psi_n(\alpha_s)|p_e|\Psi_n(\alpha_s)\rangle$, namely, the momentum expectation value on the state $|\Psi_n(\alpha_s)\rangle$. To proceed we first write Floquet eigenstates $|\psi_n(\phi, \alpha)\rangle$ as the product of two parts using the Bloch theorem, namely, $|\psi_n(\phi, \alpha)\rangle = \hat{X}(\phi)|u_n(\phi, \alpha)\rangle$, with $\hat{X}(\phi) \equiv e^{ip_e\phi/N\hbar_e}$. We then define $|\bar{u}_n(\phi, \alpha)\rangle \equiv \sum_{m=1}^{N} |m\rangle\langle m|u_n(\phi, \alpha)\rangle$. It can be shown that the N-element state $|\bar{u}_n(\phi, \alpha)\rangle$ is an eigenstate of the following $N \times N$ reduced Floquet matrix

$$\langle m|\bar{U}(\phi, \alpha)|m'\rangle \equiv \sum_{l=-\infty}^{\infty} \langle m|\hat{X}^\dagger(\phi)\hat{U}_R(\alpha)\hat{X}(\phi)|m' + lN\rangle, \qquad (4)$$

with $m, m' = 1, \cdots, N$. Lengthy but straightforward calculations [15] then yield a compact expression for $\langle p_e\rangle$, i.e., $\langle p_e\rangle = \frac{N}{2\pi}\int_0^{2\pi} d\phi \langle\bar{u}_n(\phi, \alpha_s)|i\hbar_e \frac{\partial}{\partial\phi}|\bar{u}_n(\phi, \alpha_s)\rangle$. To compare this expectation value with its preceding value for $\alpha = \alpha_{s-1}$, we consider a first-order perturbation theory so as to rewrite $|\bar{u}_n(\phi, \alpha_s)\rangle$ in terms of $|\bar{u}_n(\phi, \alpha_{s-1})\rangle$. Specifically, to the first order of $\delta\alpha \equiv \alpha_s - \alpha_{s-1} = 2\pi/s_f$, we have [15]

$$|\bar{u}_n(\phi, \alpha_s)\rangle = |\bar{u}_n(\phi, \alpha_{s-1})\rangle + \delta\alpha \sum_{n'=1,\neq n}^{N} \frac{\langle\bar{u}_{n'}(\phi, \alpha_{s-1})|\frac{\partial\bar{U}(\phi,\alpha_{s-1})}{\partial\alpha_{s-1}}|\bar{u}_n(\phi, \alpha_{s-1})\rangle}{e^{i\omega_n(\phi,\alpha_{s-1})} - e^{i\omega_{n'}(\phi,\alpha_{s-1})}}|\bar{u}_{n'}(\phi, \alpha_{s-1})\rangle. \qquad (5)$$

The change in momentum expectation value (denoted by $\delta\langle p_e\rangle_s$) over the period from $t = (s-1)T$ to $(sT)^-$ can now be calculated to the first order of $\delta\alpha$,

$$\delta\langle p_e\rangle_s = \frac{1}{2\pi}N\int_0^{2\pi} d\phi\Big\{\langle\bar{u}_n(\phi, \alpha_s)|i\hbar_e\frac{\partial}{\partial\phi}|\bar{u}_n(\phi, \alpha_s)\rangle - \langle\bar{u}_n(\phi, \alpha_{s-1})|i\hbar_e\frac{\partial}{\partial\phi}|\bar{u}_n(\phi, \alpha_{s-1})\rangle\Big\}. \qquad (6)$$

Further substituting Eq. (5) into Eq. (6) yields

$$\delta\langle p_e\rangle_s = -\frac{N}{2\pi}\int_0^{2\pi} d\phi B_n(\phi, \alpha_s)\delta\alpha, \qquad (7)$$

where $B_n(\phi, \alpha)$ is identified as the Berry curvature

$$B_n(\phi, \alpha) = i\sum_{n'=1,\neq n}^{N}\Big\{\frac{\langle\bar{u}_n|\frac{\partial\bar{U}^\dagger}{\partial\phi}|\bar{u}_{n'}\rangle\langle\bar{u}_{n'}|\frac{\partial\bar{U}}{\partial\alpha}|\bar{u}_n\rangle}{|e^{i\omega_n} - e^{i\omega_{n'}}|^2} - \text{c.c}\Big\}, \qquad (8)$$

with the explicit dependences of $|\bar{u}_n(\phi, \alpha)\rangle$, $\omega_n(\phi, \alpha)$ and $\bar{U}(\phi, \alpha)$ on α and ϕ all suppressed for brevity.

The total change in momentum expectation value from $t = 0$ to $(sT)^-$ is denoted by $\Delta_{p_e}(s)$. Because $\Delta_{p_e}(s) = \sum_{s'=1}^{s} \delta\langle p_e\rangle_{s'}$, we have

$$\Delta_{p_e}(s_f) = -M \int_0^{2\pi} d\phi \int_0^{2\pi} d\alpha B_n(\phi, \alpha) = -2\pi M C_n, \tag{9}$$

where we have used $\hbar_e = 2\pi M/N$ and C_n is exactly the Chern number of the nth Floquet band, whose surface integral expression is $C_n = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{2\pi} d\alpha B_n(\phi, \alpha)$ [15]. Thus, Eq. (9) reveals that the net change in the momentum expectation value over one adiabatic cycle of α (starting from state $|\Psi_n(\alpha = 0)\rangle$) is quantized: it should be proportional to the Chern number of the nth extended Floquet band. This is our central theoretical result.

It is necessary to numerically verify our theoretical insights above. Detailed results are shown in Fig. 4, again for the case of $\hbar_e = 2\pi/3$, with the adiabatic cycle lasting for $s_f = 100$ periods (in this case, see also Fig. 3(b) for the final momentum-space profile) or $s_f = 1000$ periods. First of all, apart from the regime of K_e values near the critical point $K_e \approx 4.2\hbar_e$ (see Fig. 2), our numerical values of $\Delta_{p_e}(s_f)/(-2\pi)$ almost perfectly match the Chern numbers. This is the case before or after the jumps of the Chern numbers. The insets of both panels depict how $\Delta_{p_e}(s)/(-2\pi)$ builds up with time and eventually reaches integer values that match the Chern numbers. In addition,

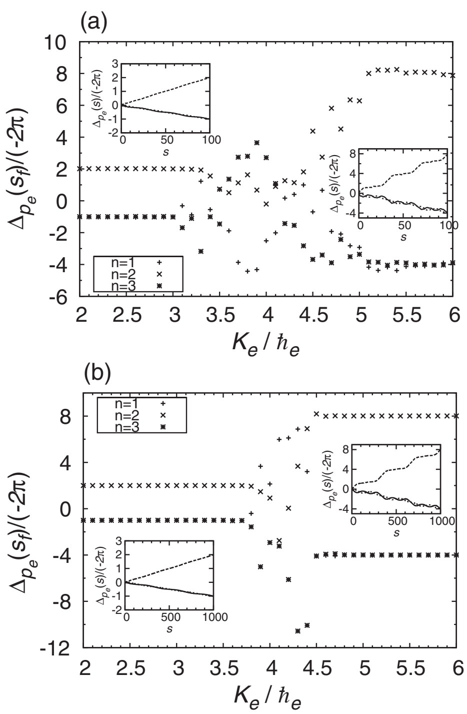

FIG. 4. Change in the momentum expectation value (divided by -2π) vs $K_e/\hbar_e$, after one adiabatic cycle implemented in (a) 100 and (b) 1000 discretized steps, for initial states prepared on each of the three Floquet bands [see Eq. (2)]. Insets shows $\Delta_{p_e}(s)/(-2\pi)$ vs number of periods s, for $K_e/\hbar_e = 2.0$ ($K_e/\hbar_e = 6.0$) on the left (right). In each inset, each of the three plotted curves is for one of the three Floquet bands, which in the end approaches integer values that match the Chern numbers.

we have checked numerically that if we repeat the adiabatic cycle, then the same quantized increase in momentum expectation value is obtained [15]. We are thus witnessing a clear quantization effect in acceleration as an outcome of Floquet band topology. Note however, in the vicinity of phase transition points, e.g., $K_e/\hbar_e \approx 4.2$, momentum-space transport is no longer quantized. This is because if a topological phase transition is about to occur, then the associated band gaps are not large enough to guarantee adiabaticity. Supporting this understanding, a comparison between Figs. 4(a) and 4(b) shows that a longer adiabatic cycle indeed significantly narrows down the nonquantization window. Our numerical data suggest that at least for the 3-band case here, if the driving field strength K_e is far away from the phase transition points, then only 50–100 kicking periods (depending on K_e) will be needed to observe quantized acceleration. This is experimentally motivating, because Floquet state manipulation itself is a topic of much theoretical interest [9,10]. The robustness of this quantization against perturbations is also examined in the Supplemental Material [15]. In short, for the 3-band case above, the quantization effect can tolerate about 0.5% uncertainty in $\hbar_e$ and a nonzero β around 0.01, which should be achievable in light of previous experiments [13].

In conclusion, based on extended Floquet bands, we have exposed topological phase transitions in driven quantum systems and demonstrated how quantized adiabatic transport in momentum space may emerge from Floquet band topology. Numerical results based on a cold-atom-based dynamical model suggest that future experimental verification of our results is possible in terms of initial state preparation, adiabatic cycle implementation, and the robustness of quantization.

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*phygj@nus.edu.sg

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