Total Nucleon-Nucleon Cross Section at Large N_c

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It is shown that at sufficiently large N_c for incident momenta which are much larger than the QCD scale, the total nucleon-nucleon cross section is independent of incident momentum and given by $\sigma^{\text{total}} = 2\pi\log^2(N_c)/(m_\pi^2)$. This result is valid in the extreme large N_c regime of $\log(N_c) \gg 1$ and has corrections of relative order $\log(\log(N_c))/\log(N_c)$. A possible connection of this result to the Froissart-Martin bound is discussed.

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The large N_c limit of QCD and the $1/N_c$ expansion have been of great interest since introduced by 't Hooft nearly 40 years ago [1]. While the approach to date has not provided a path by which quantities can be calculated *ab initio* directly from QCD, except in special cases such as QCD in $1 + 1$ dimensions [2] or QCD in the limit of heavy quark masses [3,4], it has provided a qualitative understanding of many aspects of hadronic phenomena. Witten's extension of the analysis to include baryons has played a critical role [3]. One of the remarkable features of baryons is the emergence of a contract $SU(2N_f)$ symmetry at large N_c [5] which has allowed predictions of both ground-state baryons [5] and excited baryonic resonance [6]. In this Letter, we will focus on the nonstrange sector and assume exact isospin invariance.

Implications of large N_c QCD for nuclear physics were first explored in Witten's seminal paper on large N_c baryons [3]. A key result of this analysis is that the nucleon-nucleon interaction has a strength which scales as N_c^1 and a range which scales as N_c^0. Moreover, nucleon-nucleon scattering with fixed-incident momentum has no smooth large N_c limit. However, a sensible time-dependent, mean-field description emerges if the initial velocity is held fixed at large N_c (that is, that momentum scales linearly with N_c given that the mass is linear in N_c). While there has been significant work on various aspects of nuclear physics at large N_c, such as treatments of the nucleon-nucleon (NN) potential [7], the phenomenological relevance of the large N_c limit for nuclear physics is far less clear than for hadronic physics [8]. Despite this, it is of interest to understand the N_c scaling behavior of quantities of interest in nuclear physics. One quantity that has received comparatively little attention except for a recent paper on its spin flavor dependence [9] is the total nucleon-nucleon cross section (with the effects of electromagnetic interactions removed). This is unfortunate since, as will be shown in this Letter, the total cross section is truly remarkable in that it can be computed analytically when N_c is sufficiently large:

$$\sigma^{\text{total}} = \frac{2\pi\log^2(N_c)}{m_\pi^2}. \tag{1}$$

Equation (1) holds for all spin-isopsin channels in the regime where the incident momentum is much larger than Λ_{QCD}; corrections to Eq. (1) are of relative order $\log(\log(N_c))/\log(N_c)$. Formally N_c needs to be extremely large for Eq. (1) to hold, and it is not obvious that the result is phenomenologically relevant for the physical world of $N_c = 3$. In any event, the result is of real interest from the perspective of theory.

To gain insight, it is useful to first consider a simplified "toy problem" of scattering of two nonrelativistic spinless particles of mass M interacting via a central potential that falls off exponentially at large distances. The problem has a control parameter, λ, which controls the scaling of the potential strength (but not its range), the mass, and the initial relative momentum:

$$M = \lambda\tilde{M} \qquad V(r) = \lambda\tilde{V}(r) \qquad k = \lambda\tilde{k}, \tag{2}$$

where the quantities with a tilde are independent of λ. The classical trajectory followed by a particle in this problem depends on the impact parameter b and $\tilde{p}$, $\tilde{V}(r)$, and $\tilde{M}$ but *not* on λ: if the quantum scattering is described well classically, then the differential cross section will be independent of λ. The parameter, λ, however, controls the region of validity of a semiclassical description in an underlying quantum-scattering problem; the classical limit corresponds to large λ. Of course, by design this problem mirrors the N_c scaling rules of NN scattering with λ playing the role of N_c.

The potential is central, and the partial waves are independent. Using the scaling rules in Eq. (2) and the Schrödinger equation for a given partial wave, the phase shifts can be shown to scale as

$$\delta_l(k) = \delta_{\lambda\tilde{l}}(\lambda\tilde{k}) = \lambda\tilde{\delta}_{\tilde{l}}(\tilde{k}), \tag{3}$$

where $\tilde{l} = l/\lambda$ is introduced for convenience and $\tilde{\delta}$ is independent of λ. Corrections are of relative order $1/\lambda$. To see this, parameterize the radial wave function for a given partial wave as the product of a phase and an amplitude, $\psi_l(r) = e^{i\Phi_l(r)}|\psi_l(r)|$, and take l to be proportional to λ. It is easy to show self-consistently that at large λ, Φ is proportional to λ while $|\psi_l(r)|$ is slowly varying and independent of λ. This is precisely what one expects if λ acts as the control parameter for the semiclassical limit. The total cross section for central potentials is given by

$$\sigma_{\text{toy}}^{\text{total}}(k) = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2(\delta_l(k)) \approx \frac{4\pi}{\tilde{k}^2}\int d\tilde{l}\,2\tilde{l}\sin^2(\lambda\tilde{\delta}_{\tilde{l}}(\tilde{k})), \tag{4}$$

where δ_l is the phase shift for the lth partial wave and the integral expression becomes exact as $\lambda \rightarrow \infty$.

In the integral of Eq. (4), focus on the range from $\tilde{l}_1$ to $\tilde{l}_2$. For large λ, $\sin^2$ oscillates rapidly and averages to $\frac{1}{2}$ (up to corrections of order $1/\lambda$) over this region; the contribution to the cross section becomes

$$\Delta\sigma_{\text{toy}}^{\text{total}}(k) = 2\pi(b_2^2 - b_1^2) \quad \text{with} \quad b \equiv \frac{l}{k} = \frac{\tilde{l}}{\tilde{k}}. \tag{5}$$

This is twice the geometric cross section associated with impact parameters from b_1 and b_2; the factor of 2 is due to a nearly forward diffractive scattering contribution equal to the geometrical contribution [10]. At infinite λ, there is no bound on the $\tilde{l}$s that contribute implying that the total cross section diverges as $\lambda \rightarrow \infty$.

The quantum cross section is finite because the phase shifts approach zero as $l \rightarrow \infty$. For any finite value of λ, there is a regime of sufficiently large l such that $\tilde{\delta} \sim \lambda^{-1}$ and the $\sin^2$ term does not oscillate rapidly to yield an average of $\frac{1}{2}$. The phase shift in this regime rapidly becomes small and makes small contributions to the total cross section. The value of $\tilde{l}$ beyond which the rapid oscillations effectively turns off depends on λ and gets pushed off to infinity as $\lambda \rightarrow \infty$.

To see this, start with the integral form of Eq. (4) and change variables into an integral over $\tilde{\delta}$. It is a simple matter to show that

$$\sigma_{\text{toy}}^{\text{total}} = 2\pi b_{\text{cut}}^2 + 2\pi\tilde{k}^{-2}\int_0^{\delta_{\text{cut}}/\lambda} d\tilde{\delta}\left(\frac{d\tilde{l}^2}{d\tilde{\delta}}\right)\sin^2(\lambda\tilde{\delta}) + \mathcal{O}(\lambda^{-1}), \tag{6}$$

where $\frac{d\tilde{l}^2}{d\tilde{\delta}}$ is treated as a function of $\tilde{\delta}$ in the integral and δ_{cut} is an arbitrary but fixed "cutoff" phase shift of order λ^0; b_{cut} is the impact parameter associated with δ_{cut}. The arbitrariness in the choice of δ_{cut} is compensated by the integral in Eq. (6). If the integral in Eq. (6) is parametrically smaller than $2\pi b_{\text{cut}}^2$ when δ_{cut} is of order λ^0, then up to parametrically small corrections $\sigma_{\text{toy}}^{\text{total}} = 2\pi b_{\text{cut}}^2$.

As shall be shown self-consistently, the total cross section is dominated by the behavior for large impact parameters, or equivalently, large $\tilde{l}$. In this regime, each partial wave is semiclassical and the potential is much smaller than the centrifugal barrier. Thus, the phase shifts are well approximated [11] by

$$\tilde{\delta} = -\int_{\tilde{l}/\tilde{k}}^{\infty} \frac{\tilde{\mu}\,\tilde{V}(r)}{\sqrt{\tilde{k}^2 - \tilde{l}^2/r^2}}\,dr, \tag{7}$$

where $\tilde{\mu}$ is the reduced mass divided by λ.

For concreteness, take the form of the potential to be the sum of Yukawa interactions: $\tilde{V}(r) = \sum_n \tilde{C}_n \frac{\exp(-r/r_n)}{r}$ where the $\tilde{C}_n$ are strength parameters independent of λ and the r_n are the ranges. Note that at large λ, b_{cut} is large and the integral in Eq. (7) is dominated by the longest-range contribution to the potential. Evaluating the integral yields

$$\tilde{\delta}_{\tilde{l}} = -\frac{\tilde{C}\tilde{M}}{\tilde{k}}K_0(\tilde{l}/(\tilde{k}r_0)) = -\frac{\tilde{C}_0\tilde{\mu}}{\tilde{k}}K_0(b_{\tilde{l}}/r_0), \tag{8}$$

where r_0 is the longest range in the potential and $\tilde{C}_0$ is the associated strength. For large values of $b_{\tilde{l}}/r_0$, it is legitimate to use the asymptotic form of the Bessel function when inverting this relation; doing this yields

$$b_{\tilde{l}} = \frac{r_0}{2}W\left(\frac{\tilde{C}^2\tilde{\mu}^2\pi}{\tilde{\delta}_{\tilde{l}}^2\tilde{k}^2}\right), \tag{9}$$

where W is the Lambert function. As x gets very large $W(x) \rightarrow \log(x)$ (reflecting the dominantly exponential behavior of K_0) with corrections of relative order $\log(\log(x))/\log(x)$. At large $\tilde{l}$, the phase shifts become small; the log is dominated by $\tilde{\delta}_{\tilde{l}}$; $b_l \approx -r_0\log(\tilde{\delta}_l)$ and $b_{\text{cut}} = -r_0\log(\delta_{\text{cut}}/\lambda)$. Thus, up to corrections of order λ^0,

$$b_{\text{cut}} = r_0\log(\lambda) \tag{10}$$

and at large λ is $\sigma_{\text{toy}}^{\text{total}} = 2\pi r_0^2\log^2(\lambda)$ provided the integral in Eq. (6) is parametrically small—which, as will be shown shortly, it is.

Note that the sensitivity to the particle's mass and to the strength of the potential are contained in the order λ^0 correction terms to Eq. (10). Note, further, that the sensitivity to the choice of δ_{cut} is also contained in the λ^0 correction terms. Since the dependence of the choice of δ_{cut} is compensated by the integral in Eq. (6), it follows that the integral is also parametrically of order λ^0 and makes a negligible contribution to the cross section at large λ. Thus, $\sigma_{\text{toy}}^{\text{total}} = 2\pi r_0^2\log^2(\lambda)$ with corrections of relative order $\log(\log(x))/\log(x)$. With the substitutions $r_0 \rightarrow 1/m_\pi$ and $\lambda \rightarrow N_c$, this is of the form of Eq. (1). It should be apparent that any power law prefactor to the Yukawa potentials cannot alter this result at leading order.

The result also holds for a relativistic version of the toy problem. Consider potential scattering in a relativistic two-body model (which lacks micro causality but can be consistently formulated as a quantum theory [12]). The basic setup remains intact: the partial wave decomposition still holds and Eqs. (4) and (6) remain valid. In the semiclassical regime with sufficiently large $\tilde{l}$, one can always cast the phase shift into the form of Eq. (7) with $\tilde{V}(r)$ replaced by (an energy dependent) $\tilde{V}_{\rm eff}(r)$ whose form depends on the transformation properties of the interaction. If the longest-range interaction in the model transforms as a Lorentz scalar (as in QCD), then at long range $\tilde{V}_{\rm eff}(r) = \tilde{V}_s(r)\tilde{E}/\tilde{\mu}$. Relativity affects $b_{\rm cut}$ only by renormalizing the strength of the longest-range interaction by an energy-dependent amount independent of λ. Since the leading behavior of the total cross section does not depend on the strength, the relativistic toy model also has $\sigma^{\rm total}_{\rm rel.toy} = 2\pi r_0^2 \log^2(\lambda)$ which corresponds to Eq. (1).

Nucleon-nucleon scattering in large N_c QCD is clearly more complicated than in the toy problem for several reasons: (i) The nucleons have spin, and the partial wave expansion for elastic scattering is necessarily of a coupled channel form; (ii) there is an emergent spin-isospin symmetry at large N_c [5]; as a result of this symmetry the Δ and a whole tower of baryons are stable and nearly degenerate with the nucleon at large N_c. The emergent symmetry implies correlations between channels in scattering [7,9,13]; (iii) There are inelastic channels due to meson production. However, as discussed below, the result in Eq. (1) is quite robust and is unaltered by these complications.

To treat the full problem, the generically strong (order N_c) nature of the NN interaction must be encoded in a model-independent way. The potential model treatments of the toy problem are inappropriate to this problem and, in any event, the potential is intrinsically unphysical which can lead to subtleties in the N_c counting [14]. The physically relevant object is the S matrix for nucleon-nucleon scattering, $S^{\rm NN}$. Its elements for elastic scattering can be denoted $S^{\rm NN}_{l,a;l',a'}$ where a (a') specifies the spin and isospin configuration of the incident (final) states of the two nucleons. Conservation of angular momentum and isospin along with the fermion nature of nucleons constrain the form of $S^{\rm NN}$; for example, the matrix elements are zero unless $l' = l-1, l, l+1$.

The strong NN interaction at large N_c cannot be encoded by having $S^{\rm NN}$ scale linearly in N_c; it is bounded due to unitarity. To proceed, note that in the toy problem, Eq. (3), the phase shift, i.e., the logarithm of the S matrix in the partial wave channel, scales with N_c. This behavior is expected to hold generically for diagonal matrix elements of the S matrix in large N_c QCD.

$$\log(S^{\rm NN}_{l,a;la}) \equiv 2i\delta_{l,a} = 2i\delta^R_{l,a} - 2\delta^I_{l,a} \sim N_c \qquad (11)$$

where the S matrix is for nucleon-nucleon elastic scattering at fixed initial velocity. Note that $\delta_{l,a}$ is not real in general; the imaginary part reflects scattering out of the original channel either to other elastic channels (with different final l or a) or to inelastic channels. Both the real and imaginary parts are expected to scale with N_c. This applies to all physical channels (e.g., two neutrons with spins aligned with the beam).

There are several ways to understand the origin of the scaling in Eq. (11); the simplest is via an optical potential for relativistic nucleon-nucleon scattering; its imaginary part encodes loss of flux into channels with particle creation. Using Witten's counting rules, one sees that the generic counting for both the real and the imaginary parts of the optical potential both scale with N_c. Semiclassical analysis analogous to the derivation of Eq. (7), then, straightforwardly yields Eq. (11). A more complete derivation of Eq. (11) will be discussed in a forthcoming publication.

The total cross section in multichannel problems with the initial nucleons in spin-isospin configuration a can be expressed in terms of the diagonal matrix elements of the S matrix:

$$\sigma^{\rm total}_a(k) = \frac{2\pi}{k^2}\sum_l (2l+1)\{1-\exp[-2\delta^I_{l,a}(k)]\cos[2\delta^R_{l,a}(k)]\}$$
$$\approx \frac{4\pi}{\tilde{k}^2}\int d\tilde{l}^2\{1-\exp[-2N_c\delta^I_{\tilde{l},a}(N_c\tilde{k})]$$
$$\times\cos[2N_c\delta^R_{\tilde{l},a}(N_c\tilde{k})]\}, \qquad (12)$$

where $k = N_c\tilde{k}$; the first form is general [11] and the second form builds in the N_c scaling of the phase shifts. The integral form becomes exact in the limit $N_c \to \infty$.

Note that Eq. (12) coincides with Eq. (4) of the toy problem if one sets $\delta^I_{l,a}$ to zero. Moreover, Eq. (5) continues to hold, providing the integrand is in the regime where either the real or imaginary parts of $\tilde{\delta}$ (or both) are of order unity; in the case of the real part, it is due to rapid oscillations as in the toy problem; in the case of the imaginary part, it holds due to an exponential suppression. As in the toy problem, the total cross section is determined by where $\tilde{\delta}$ ceases to be of order unity and becomes of order $1/N_c$. It is easy to see that if, as a function of l, $\delta^I_{l,a}$ approaches zero at least as rapidly as $\delta^R_{l,a}$, then the total cross section at leading order will be determined by where $\tilde{\delta}^R_{\tilde{l},a}$ drops to order of $1/N_c$.

It is clear that $\delta^I_{l,a}$ *does* approach zero more rapidly than $\delta^R_{l,a}$: consider going to sufficiently large $\tilde{l}$ so that the phase shifts are accurately described by the Born approximation for the longest range part of the interaction—one-pion exchange. In that regime, the $\delta^R_{l,a}$ is small but nonvanishing, while $\delta^I_{l,a}$ vanishes at the first Born approximation level only arising at second order. Moreover, it is clear that at very large l, the real part of the phase shift is dominated by the Born approximation contribution to one-pion exchange which drops off exponentially in exactly the same way that it does in the toy model. Accordingly, the result in the toy model carries across, and Eq. (1) follows exactly as in the toy model. Note that this exponential falloff holds for any physical spin and isospin configuration of the initial baryons; pion exchange dominates regardless of the initial spin orientations or whether the two nucleons are the same or different. Thus Eq. (1) also holds for any initial configuration, and the leading order cross section is spin and isospin independent. This is consistent with the analysis of Ref. [9], although it is more restrictive than the most general leading-order result deduced there.

It is worth observing that at large N_c, elastic scattering should account for half of the total scattering or more. In the regime of l, where $\delta^I_{l,a} \sim N_c$, the contribution to the scattering looks like a black disk for which diffractive scattering is 50%. There may, in principle, also be substantial contributions for the regime where $\delta^I_{l,a}$ is small but $\delta^R_{l,a} \sim N_c$; such contributions will have elastic contributions of greater than 50%.

The $\log^2$ form of the cross section in Eq. (1) is strikingly similar to the Froissart-Martin bound [15]. At large Mandelstam s, considerations of unitarity, analyticity, plus the knowledge that the pion is the lightest excitation in the system serve to bound the growth of the total cross section at large s

$$\sigma^{\rm total} \leq \frac{\pi}{m_\pi^2}\log^2\left(\frac{s}{s_0}\right), \qquad (13)$$

where s_0 is a reference scale. This similarity may not be accidental. Note that the natural regime for nucleon-nucleon scattering at large N_c is for fixed velocity [3], which in turn implies that $s \sim N_c^2$. Provided that s_0 does not also scale with N_c, the bound becomes $\sigma^{\rm total} \leq \frac{\pi}{m_\pi^2}\log^2(\frac{N_c\tilde{s}}{s_0})$, where $\tilde{s}$ is independent of N_c. If one takes the large N_c limit prior to the large s limit, and keeps only the leading behavior, one has $\sigma^{\rm total} \leq \frac{4\pi}{m_\pi^2}\log^2(N_c)$ up to corrections of relative order $1/\log(N_c)$. Note that Eq. (1) satisfies this inequality by exactly a factor of $\frac{1}{2}$. This factor of $\frac{1}{2}$ is suggestive. The derivation of the Froissart-Martin bound requires unitarity. However, if one looks at the integral form of Eq. (12) it is clear that in the region of dominant contribution, the integrand at large N_c is precisely $\frac{1}{2}$ of its unitarity bound (which occurs at $\delta^R = \pi/2$, $\delta^I = 0$). Thus, the present result is natural in light of the Froissart-Martin bound.

To what extent is this result applicable to the physical world of $N_c = 3$? In the physical world, the total cross section for $\sqrt{s}$ well above ΛQCD is approximately independent of s [16] as would be expected from the large N_c analysis. Over three orders of magnitude in $\sqrt{s}$, from 1.5 GeV $< \sqrt{s} <$ 1200 GeV, the cross section for proton-proton scattering varies by only about 25%. Moreover, the cross section is dominantly spin and isospin independent [9,17] as predicted to occur at large N_c. These results may suggest that the large N_c analysis is of phenomenological relevance for the physical world of $N_c = 3$. However, this is not clear. For example, above 1.5 GeV, the total cross section is predominantly inelastic; the elastic cross section is typically less than $\frac{1}{4}$ of the total cross section and by $\sqrt{s}$ of several 10 s of GeV, it drops to under 20%. While it has been argued that at truly asymptotically high energies [18] it approaches $\frac{1}{2}$, at large N_c this behavior is expected for all s well above $\Lambda_{|rm{\rm QCD}}$. This implies that a substantial part of the cross section comes from regions where both the real and imaginary parts of the phase shift are small. This is at odds with the behavior expected at large N_c, where, as noted above, elastic scattering should be 50% or higher. Given this, it is perhaps not too surprising that the absolute prediction of Eq. (1) that $\sigma^{\rm total} \approx 150$ mb is significantly larger than the empirical value of approximately 40 mb. Ultimately, the reason that the large N_c analysis for the magnitude of the total cross sections is not very predictive for the $N_c = 3$ world is quite understandable. Formally, one expects the analysis to be predictive only when $\log(N_c) \gg 1$. Clearly this is not true for $N_c = 3$. Whatever the phenomenological significance for the world of $N_c = 3$, the fact that at large N_c the total cross section is calculable is, at the very least, of theoretical interest.

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[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[3] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[4] T. D. Cohen, N. Kumar, and K. K. Ndousse, Phys. Rev. C **84**, 015204 (2011).
[5] J. L. Gervais and B. Sakita, Phys. Rev. Lett. **52**, 87 (1984); R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. D **49**, 4713 (1994);C. Carone, H. Georgi, and S. Osofsky, Phys. Lett. B **322**, 227 (1994);M. Luty and J. March-Russell, Nucl. Phys. **B426**, 71 (1994).
[6] T. D. Cohen and R. F. Lebed, Phys. Rev. Lett. **91**, 012001 (2003); T. D. Cohen and R. F. Lebed, Phys. Rev. D **67**, 096008 (2003);T. D. Cohen and R. F. Lebed, Phys. Rev. D **68**, 056003 (2003).
[7] D. B. Kaplan and M. J. Savage, Phys. Lett. B **365**, 244 (1996); D. B. Kaplan and A. V. Manohar, Phys. Rev. C **56**, 76 (1997); T. D. Cohen and B. A. Gelman, Phys. Lett. B **540**, 227 (2002); M. K. Banerjee, T. D. Cohen, and B. A. Gelman, Phys. Rev. C **65**, 034001 (2002).
[8] F. Gross, T. D. Cohen, E. Epelbaum, and R. Machleidt, Few-Body Syst. **50**, 31 (2011).
[9] T. D. Cohen and B. A. Gelman, Phys. Rev. C **85**, 024001 (2012).
[10] J. D. Jackson, *Classical Electrodynamics* (Wiley , New York, 1998), 3rd ed.
[11] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Butterworth-Heinemann, Washington, DC, 1981), 3rd ed..
[12] W. N. Polyzou, Y. Huang, Ch. Elster, W. Glockle, J. Golak, R. Skibinski, H. Witala, and H. Kamada, Few-Body Syst. **49**, 129 (2011).

[13] T. D. Cohen and B. A. Gelman, Phys. Lett. B **540**, 227 (2002).

[14] A. V. Belitsky and T. D. Cohen, Phys. Rev. C **65**, 064008 (2002); T. D. Cohen, Phys. Rev. C **66**, 064003 (2002).

[15] M. Froissart, Phys. Rev. **123**, 1053 (1961); A. Martin, Phys. Rev. **129**, 1432 (1963).

[16] K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

[17] C. Lechanoine-LeLuc and F. Lehar, Rev. Mod. Phys. **65**, 47 (1993).

[18] M. M. Block and F. Halzen, Phys. Rev. Lett. **107**, 212002 (2011).