P H Y S I C A L R E V I E W L E T T E R S

Quantum Measurement Bounds beyond the Uncertainty Relations

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In quantum mechanics, the Heisenberg uncertainty relations and the Cramér-Rao inequalities typically limit the precision in the estimation of a parameter through the standard deviation of a conjugate observable. Here we extend these relations by giving a bound to the precision of a parameter in terms of the expectation value of the conjugate observable. This has both foundational and practical consequences: in quantum optics it resolves a controversy over which is the ultimate precision in interferometry.

Quantum mechanics limits the accuracy with which one can measure conjugate quantities: the Heisenberg uncertainty relations [1,2] and the quantum Cramér-Rao inequality [3–6] show that no procedure for estimating the value of some quantity (e.g., a relative phase) can have a precision that scales more accurately than the inverse of the standard deviation of a conjugate quantity (e.g., the energy) evaluated on the state of the probing system. This paper exhibits a new bound on quantum measurement: we prove that the precision of measuring a quantity cannot scale better than the inverse of the expectation value (above a "ground state") of the conjugate quantity. We use the bound to resolve an outstanding problem in quantum metrology [7]: in particular, we prove the longstanding conjecture of quantum optics [8–13]—recently challenged [14–16]—that the ultimate limit to the precision of estimating phase in interferometry is bounded below by the inverse of the total number of photons employed in the interferometer.

The statistical nature of quantum mechanics induces fluctuations that limit the ultimate precision that can be achieved in the estimation of any parameter x. These fluctuations can be connected to the properties of a conjugate operator H that generates translations $U_x = e^{ixH}$ of the parameter x. In particular, if the encoding stage is repeated several times using ν identical copies of the same probe input state ρ_x, the root mean square error (RMSE) ΔX of the resulting estimation process is limited by the quantum Cramér-Rao bound [3–6] $\Delta X \geq 1/\sqrt{\nu Q(x)}$, where $Q(x)$ is the quantum Fisher information. For pure probe states and unitary encoding mechanism U_x, $Q(x)$ is equal to the variance $(\Delta H)^2$ (calculated on the probe state) of the generator H. In this case, the Cramér-Rao bound takes the form

$$\Delta X \geq 1/(2\sqrt{\nu}\Delta H) \qquad (1)$$

of an uncertainty relation [5,6]. This bound is asymptotically achievable in the limit of $\nu \to \infty$ [3,4]. If the parameter x can be connected to an observable, Eq. (1)

corresponds to the Heisenberg uncertainty relation for conjugate variables [1,2]. (Note that we can also exploit quantum "tricks" such as entanglement and squeezing in optimizing the state preparation of the probe and/or the detection stage [17].)

Here we will derive a bound in terms of the expectation value $\langle H \rangle$ of H, which in the simple case of constant ΔX, takes the form (see Fig. 1)

$$\Delta X \geq \kappa/[\nu(\langle H \rangle - E_0)], \qquad (2)$$

where E_0 is the value of a "ground state," the minimum eigenvalue of H whose eigenvector is populated in the
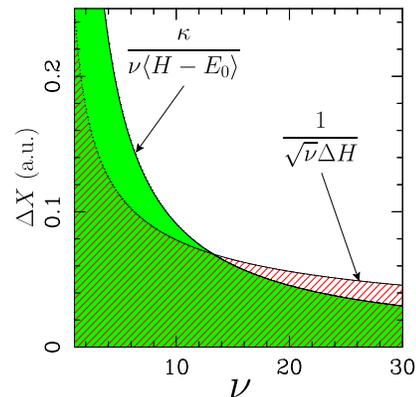
FIG. 1 (color online). Lower bounds to the precision estimation ΔX as a function of the experimental repetitions ν. The gray area in the graph represents the forbidden values due to our bound (2). The hatched (dashed-line) area represents the forbidden values due to the Cramér-Rao bound, or the Heisenberg uncertainty, (1). Possible estimation strategies have precision ΔX that cannot penetrate in the colored regions. For large ν the Cramér-Rao bound (which scales as $1/\sqrt{\nu}$) is stronger, as expected since in this regime it is achievable. Our bound is not achievable in general, so that the gray area may be expanded when considering specific estimation strategies. [Here we used $\langle H \rangle - E_0 = 0.1$ (a.u.) and $\Delta H = 4$ (a.u.).]

probe state (e.g., the ground state energy when H is the probe's Hamiltonian), and $\kappa \simeq 0.091$ is a constant of order one. We stress that, because of the presence of the factor ν, the quantity at the denominator of (2) is associated to the global generator of translations of the ν copies. The inequality (2) holds both for biased and unbiased measurement procedures, for pure and mixed probe states, and it is consistent with the recent bounds [18] on the number of distinguishable states crossed by the evolution U_x. As discussed in the following, the bound holds for "good" estimation strategies that provide enough information on the parameter x [19]. Note also that the rhs of (2) diverges for $\langle H \rangle \to E_0$ because, if the probe is in the ground state of the generator H, its final state is independent of x and provides no information on it.

The bound (2) must be modified for procedures where ΔX explicitly depends on x, as in the examples discussed in Refs. [14,15]: a constraint of the form (2) is placed on the average value of $\Delta X(x)$. Specifically given any two values x and x' of the parameter which are sufficiently separated, one has

$$\frac{\Delta X(x) + \Delta X(x')}{2} \geq \frac{\kappa}{\nu(\langle H \rangle - E_0)}. \qquad (3)$$

Hence, even though we cannot exclude that strategies whose error ΔX depend on x may have a "sweet spot" where the bound (2) may be beaten [14,15], Eq. (3) shows that the average value of ΔX is subject to a bound that scales as the inverse of $\nu(\langle H \rangle - E_0)$. As a consequence, these strategies are of no practical use since the sweet spot depends on the unknown parameter x to be estimated and the extremely good precision in the sweet spot must be counterbalanced by a correspondingly bad precision nearby.

Proving Eq. (2) in full generality is clearly not a trivial task since no definite relation can be established between $\nu(\langle H \rangle - E_0)$ and the term $\sqrt{\nu} \Delta H$ on which the Cramér-Rao bound is based. In particular, scaling arguments on ν cannot be used since, on one hand, the value of ν for which Eq. (1) saturates is not known (except in the case in which the estimation strategy is fixed [8], which has little fundamental relevance) and, on the other hand, input probe states ρ whose expectation values $\langle H \rangle$ depend explicitly on ν may be employed, e.g., see Ref. [15]. To circumvent these problems our proof is based on the quantum speed limit [21], a generalization of the Margolus-Levitin [22] and Bhattacharyya bounds [23,24] which links the fidelity F [25] between the two joint states $\rho_x^{\otimes \nu}$ and $\rho_{x'}^{\otimes \nu}$ to the difference $x' - x$ of the parameters x and x' imprinted on the states through the mapping $U_x = e^{-ixH}$ [26]. In the case of interest here, the quantum speed limit [21] implies

$$|x' - x| \geq \frac{\pi}{2} \max \left[\frac{\alpha(F)}{\nu(\langle H \rangle - E_0)}, \frac{\beta(F)}{\sqrt{\nu} \Delta H} \right], \qquad (4)$$

where the ν and $\sqrt{\nu}$ factors at the denominators arise from the fact that here we are considering ν copies of the

probe states ρ_x and $\rho_{x'}$, and where $\alpha(F) \simeq \beta^2(F) = 4\arccos^2(\sqrt{F})/\pi^2$ are decreasing functions defined in [21]. The inequality (4) tells us that the parameter difference $|x' - x|$ induced by a transformation $e^{-i(x'-x)H}$ that employs resources $\langle H \rangle - E_0$ and ΔH cannot be arbitrarily small (when the parameter x coincides with the evolution time, this sets a limit to the "speed" of the evolution, the quantum speed limit).

We now give the main ideas of the proof of (2) by focusing on a simplified scenario, assuming pure probe states $|\psi_x\rangle = U_x|\psi\rangle$, and unbiased estimation strategies constructed in terms of projective measurements with RMSE ΔX that do not depend on x. The detailed proof is given in [27], where these restrictions are dropped. For unbiased estimation, $x = \sum_j P_j(x)x_j$ and the RMSE coincides with the variance of the distribution $P_j(x)$, i.e., $\Delta X = \sqrt{\sum_j P_j(x)[x_j - x]^2}$, where $P_j(x) = |\langle x_j | \psi_x \rangle^{\otimes \nu}|^2$ is the probability of obtaining the result x_j while measuring the joint state $|\psi_x\rangle^{\otimes \nu}$ with a projective measurement on the joint basis $|x_j\rangle$. Let us consider two values x and x' of the parameter that are further apart than the measurement's RMSE, i.e., $x' - x = 2\lambda \Delta X$ for a $\lambda \gtrsim 1$ that will be specified later. If no such x and x' exist, the estimation is extremely poor: basically the whole domain of the parameter is smaller than (or of the same order of) the RMSE. Hence, we can always assume that such a choice is possible for estimation strategies that are sufficiently accurate to be of interest, as discussed in detail below. The Chebyshev inequality states that for an arbitrary probability distribution p, the probability that a result x lies more than $\lambda \Delta X$ away from the average μ is upper bounded by $1/\lambda^2$, namely $p(|x - \mu| \geq \lambda \Delta X) \leq 1/\lambda^2$. It implies that the probability that measuring $|\Psi_{x'}\rangle := |\psi_{x'}\rangle^{\otimes \nu}$ the outcome x_j lies within $\lambda \Delta X$ of the mean value associated with $|\Psi_x\rangle := |\psi_x\rangle^{\otimes \nu}$ cannot be larger than $1/\lambda^2$. By the same reasoning, the probability that measuring $|\Psi_x\rangle$ the outcome x_j will lie within $\lambda \Delta X$ of the mean value associated with $|\Psi_{x'}\rangle$ cannot be larger than $1/\lambda^2$. This implies that the overlap between the states $|\Psi_x\rangle$ and $|\Psi_{x'}\rangle$ cannot be too large: more precisely $F = |\langle \Psi_x | \Psi_{x'} \rangle|^2 \leq 4/\lambda^2$. Replacing this expression into (4) (exploiting the fact that α and β are decreasing functions) we obtain

$$2\lambda \Delta X \geq \frac{\pi}{2} \max \left[\frac{\alpha(4/\lambda^2)}{\nu(\langle H \rangle - E_0)}, \frac{\beta(4/\lambda^2)}{\sqrt{\nu} \Delta H} \right], \qquad (5)$$

whence we obtain (2) by optimizing over λ the first term of the max, i.e., choosing $\kappa = \sup_\lambda \pi \alpha(4/\lambda^2)/(4\lambda) \simeq 0.091$, maximized for $\lambda \simeq 4.09$. The second term of the max gives rise to a quantum Cramér-Rao type uncertainty relation (or a Heisenberg uncertainty relation) that, consistently with the optimality of Eq. (1) for $\nu \gg 1$, has a pre-factor $\pi \beta(4/\lambda^2)/(4\lambda)$ that is smaller than $1/2$ for all λ. This means that for large ν the bound (2) will be asymptotically

superseded by the Cramér-Rao part, which scales as $\propto 1/\sqrt{\nu}$ and is achievable in this regime. In other words, when it is physically significant, the Cramér-Rao bound always wins over the bound (2).

Analogous results can be obtained (see [27]) when considering more general scenarios where the input states of the probes are not pure, the estimation process is biased, and it is performed with arbitrary (possibly adaptive) positive operator-valued measure (POVM) measurements. [In the case of biased measurements, the constant κ in (2) and (3) must be replaced by $\kappa = \sup_\lambda \pi\alpha(4/\lambda^2)/[4(\lambda + 1)] \simeq 0.074$ (maximized for $\lambda \simeq 4.64$) where a $+1$ term appears in the denominator.] In this generalized context, whenever the RMSE depends explicitly on the value x of the parameter, the result (2) derived above is replaced by the weaker relation (3). Such inequality clearly does not necessarily exclude the possibility that at a "sweet spot" the estimation might violate the scaling (2), as happens, e.g., in the strategies of [14,15], which are, hence, fully compatible with our bounds. However, Eq. (3) is still sufficiently strong to exclude accuracies of the form $\Delta X(x) = 1/R(x, \nu\langle H\rangle)$ where, as in Refs. [15,28], $R(x, z)$ is a function of z that, for all x, increases more than linearly, i.e., $\lim_{z\to\infty} z/R(x, z) = 0$.

The bound (2) has been derived under the explicit assumption that x and x' exists such that $x' - x = 2\lambda\Delta X$ for some $\lambda \simeq 4.09$, which requires one to have $x' - x \geq 2\Delta X$. This means that the estimation strategy must be good enough: the probe is sufficiently sensitive to the transformation U_x that it is shifted by more than ΔX during the interaction. The existence of pathological estimation strategies which violate such condition cannot be excluded *a priori*. Indeed trivial examples of this sort can be easily constructed, a fact which may explain the complicated history of the Heisenberg bound with claims [8–13] and counterclaims [14–16,28]. It should be stressed, however, that the assumption $x' - x \geq 2\Delta X$ is always satisfied except for extremely poor estimation strategies with such large errors as to be practically useless. One may think of repeating such a poor estimation strategy $\nu > 1$ times and of performing a statistical average to decrease its error. However, for sufficiently large ν the error will decrease to the point in which the ν repetitions of the poor strategy are, collectively, a good strategy and hence, again subject to our bounds (2) and (3).

Our findings are particularly relevant in the field of quantum optics, where a controversial and long-debated problem [8–16,28] is to determine the scaling of the ultimate limit in the interferometric precision of estimating a phase as a function of the total average energy devoted to preparing the ν copies of the probes: it has been conjectured [8–13] that the phase RMSE is lower bounded by the inverse of the total number of photons employed in the experiment, the "Heisenberg bound" for interferometry [29]. It corresponds to an equation of the form of Eq. (2),

choosing $x = \phi$ (the relative phase between the modes in the interferometer) and $H = a^\dagger a$ (the number operator). Scalings of this sort have been established for some redefinitions of the uncertainty measure [3] or for specific detection strategies [30] (see, e.g., Refs. [31–33] and references therein) while its achievability for the RMSE measure has been recently proven in [34]. Still its general validity for the RMSE has been questioned several times [14–16,28]. In particular schemes have been proposed [15,28] that apparently permit better scalings in the achievable RMSE (for instance $\Delta X \approx (\nu\langle H\rangle)^{-\gamma}$ with $\gamma > 1$). None of these protocols have conclusively proved such scalings for arbitrary values of the parameter x, but a sound, clear argument against the possibility of breaking the $\gamma = 1$ scaling of Eq. (2) was missing up to now. Our results validate the Heisenberg bound by showing that it applies to all those estimation strategies whose RMSE ΔX does not depend on the value of the parameter x and that the remaining strategies can have good precision only for isolated values of the unknown parameter x.

After the appearance of the first version of our manuscript, related papers have appeared in which the bounds derived here are analyzed in the presence of nontrivial prior information [20,35]. Moreover, for optical interferometry our findings have been strengthened in Ref. [36] and a compatible bound dependent on the prior information was obtained through rate distortion theory in [37].

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