Laughlin Spin-Liquid States on Lattices Obtained from Conformal Field Theory

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We propose a set of spin system wave functions that is very similar to lattice versions of the Laughlin states. The wave functions are conformal blocks of conformal field theories and for a filling factor of $\nu = 1/2$ we provide a parent Hamiltonian, which is valid for any even number of spins and is at the same time a 2D generalization of the Haldane-Shastry model. We also demonstrate that the Kalmeyer-Laughlin state is reproduced as a particular case of this model. Finally, we discuss various properties of the spin states and point out several analogies to known results for the Laughlin states.

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Phenomena in strongly correlated systems are generally hard to understand and describe, and therefore, simple model systems exhibiting various behaviors are important guides. Laughlin's wave functions [1]

$$\prod_n \tilde{\chi}(Z_n) \prod_{n<p} (Z_n - Z_p)^{1/\nu} \exp\left(-\sum_q |Z_q|^2/4\right), \qquad (1)$$

e.g., have played a key role in explaining the fractional quantum Hall effect, and this has triggered interest in understanding the nature of these states in depth. [The complex numbers Z_n are the positions of the particles in the complex plane, ν is the Landau level filling factor, and the product of the single-particle phase factors $\tilde{\chi}(Z_n)$ is the gauge factor.] Finding parent Hamiltonians for wave functions is also very useful because it tells us how the behavior described by the wave function can be associated to the interactions between the particles in the system, and it can guide us to find experimental situations where such a behavior occurs, even if the Hamiltonian itself cannot be implemented directly.

In 1987, Kalmeyer and Laughlin (KL) introduced the $\nu = 1/2$ bosonic Laughlin state on a square lattice [2]. Such models are expected to play a crucial role in understanding topological phases of lattice systems, in very much the same way as Laughlin states do in bulk. Furthermore, they may open up the door [3] for the experimental realization and investigation of Laughlin-like states under very well-controlled conditions in optical lattices. The KL state has been further investigated in [4,5], but it has been a long-standing problem to find a parent Hamiltonian for the state. Within the last few years, Hamiltonians have been found that are exact in the thermodynamic limit [6–8]. In this Letter, we take a different approach in which we propose to slightly modify the Laughlin states. The modification enables us to find a relatively simple Hamiltonian, containing only two- and three-body interactions, which is exact also for finite systems and arbitrary lattices. We demonstrate that the modified states are very close to the original Laughlin lattice states for finite systems, while they are exactly the same in the limit of an infinite square lattice.

As in the spirit of [9], the wave functions we propose are chiral correlators of conformal blocks. The key element in the derivation of the Hamiltonian is to exploit that this structure allows us to apply rules from conformal field theory (CFT). Using the properties of null fields in CFT, we have recently derived [10] nonuniform and higher spin generalizations of the 1D Haldane-Shastry (HS) model [11,12], and in this Letter, we extend these results further to obtain a generalization of the HS model to 2D. Previous work on finding parent Hamiltonians for the KL state on an infinite lattice has in part been inspired by the original HS model. Here, we complete this idea by demonstrating that the HS state and the KL state are in fact two limiting cases of the same model.

In addition, we characterize the most important physical properties of the modified states, including correlation functions, topological entanglement entropies (EEs), and entanglement spectra. It has been noted numerically that for some fractional quantum Hall states, the entanglement spectrum corresponds to the spectrum of the CFT that defines the wave function on the boundary [13]. Utilizing the particular structure of the proposed wave functions, we are here able to show analytically that the entanglement spectrum for a two-legged ladder (as defined in [14]) exactly corresponds to the 1D CFT with central charge $c = 1$ for $\nu = 1/2$, and we trace this back to the fact that the Yangian symmetry is inherited at the boundary.

Wave function.—The wave functions we propose describe the state of N spin 1/2 particles at positions $z_1, \ldots, z_N$ in the complex plane, where N is even. They are chiral correlators of products of vertex operators $\phi_{s_n}(z_n) =: e^{i\sqrt{\alpha}s_n\varphi(z_n)}:$ [15], where $:\ldots:$ means normal ordering, α is a positive parameter, $s_n = \pm 1$ is twice the z component of the nth spin, and $\varphi(z_n)$ is the field of a free mass-less boson, i.e., [15],

$$\psi_{s_1,\ldots,s_N}(z_1,\ldots,z_n) = \langle \phi_{s_1}(z_1)\cdots\phi_{s_N}(z_N)\rangle = \delta_{\mathbf{s}} \prod_{p=1}^{N} \chi_{p,s_p} \prod_{n<m}^{N} (z_n - z_m)^{\alpha s_n s_m}. \quad (2)$$

Here, $\delta_{\mathbf{s}} = 1$ for $\sum_n s_n = 0$ and $\delta_{\mathbf{s}} = 0$ otherwise, and the phase factors χ_{p,s_p} can be chosen at will, since the chiral correlator is only defined up to a phase. For $\alpha = 1/2$, we shall always choose $\chi_{p,s_p} = \exp[i\pi(p-1)(s_p+1)/2]$, since this ensures that (2) is a singlet [16].

For $\alpha = 1/4$, we note that (2) can be written (up to an overall phase) as a Slater determinant of the single-particle wave functions $\psi_k(z_n) = z_n^{k-1}(\chi_{n,1}/\chi_{n,-1})\prod_{m(\neq n)}\times (z_n - z_m)^{-1/2}$, $k = 1, 2, \ldots, N/2$. [We use the convention that $\prod_{n\neq m}$ is the product over m and n, whereas $\prod_{m(\neq n)}$ is the product only over m.] The states $\psi_k(z_n)$ are not orthonormal, but can be made so without changing the Slater determinant. We can therefore regard (2) as the state of $N/2$ noninteracting fermions. This simplification enables us to use exact numerical computations rather than Monte Carlo simulations for $\alpha = 1/4$ when we compute properties of the wave functions below. The $\nu = 1$ Laughlin state can also be written as a Slater determinant, and indeed we shall see in a moment that $\alpha = 1/4$ corresponds to $\nu = 1$.

Connection to the Laughlin states.—We next investigate the statement that (2) is similar to lattice versions of the Laughlin states, which are in turn closely related to the continuous Laughlin states. We expect the correspondence to be approximately valid for all lattice configurations for which the distribution of the lattice points is not too far from uniform and also if the complex plane is mapped into other geometries. We shall here consider the case of a (finite) square lattice in the complex plane since it is mathematically convenient and the case of an approximately uniform distribution on the sphere because this geometry eliminates all boundaries.

It has been speculated in [17] that the KL state is proportional to the conformal block in (2) with $\alpha = 1/2$ on an infinite square lattice. Here, we proof explicitly in the Supplemental Material [16] for the case of a $2M \times 2M$ square lattice centered at the origin and with lattice constant $b = \sqrt{4\pi\alpha}$ that the ratio between (2) with appropriately chosen χ_{p,s_p} and (1) with $\nu = (4\alpha)^{-1}$ is given by

$$\prod_{n=1}^{4M^2} |f_M(z_n)/f_\infty(z_n)|^{\alpha(1+s_n)} \quad (3)$$

up to an irrelevant overall factor, where $f_M(z_n) \equiv (z_n/b)\times \prod_{m(\neq n)}^{4M^2}(1 - z_n/z_m)^{-1}$. In particular, the two wave functions coincide for $M \to \infty$. In brief, we prove this result by transforming the spins into hard-core bosons by writing $s_n = 2q_n - 1$, $q_n \in \{0, 1\}$. This allows us to express (2) in terms of $f_M(z_n)$. We then take the limit $M \to \infty$, compute $f_\infty(z_n)$ by algebraic methods, and compare the result to (1).

We note that the correct density of particles is obtained by scaling the lattice constant rather than by changing the filling factor of the lattice. As a consequence, the lattice filling factor, which is always $1/2$, only coincides with the Landau level filling factor ν for $\alpha = 1/2$. Figure 1(a) demonstrates that $|f_M(z)| \approx |f_\infty(z)|$ already for a 10×10 lattice, and thus there is a close relationship between Eqs. (1) and (2) even for small systems.

The model can be mapped from the complex plane to the unit sphere with polar angle θ and azimuthal angle ϕ by the stereographic projection $z = v/u$, where $u = \cos(\theta/2)e^{i\phi/2}$ and $v = \sin(\theta/2)e^{-i\phi/2}$. A small computation shows that $\rho_{ij} \equiv (v_i u_j - u_i v_j)^{-1}$ and also the wave function (2) are invariant under $SU(2)$ transformations of the pair (u, v). Note that $d_{ij} = 2|\rho_{ij}|^{-1}$ is the shortest distance $d_{ij} = |\mathbf{n}_i - \mathbf{n}_j|$ between spin i and spin j, where $\mathbf{n}_i \equiv (\sin(\theta_i)\cos(\phi_i), \sin(\theta_i)\sin(\phi_i), \cos(\theta_i))$ is the position vector of spin i. Writing $s_n = 2q_n - 1$ as before, we find that (2) is proportional to the Laughlin wave function on the sphere $\prod_{n<m}^{N} \rho_{nm}^{-q_n q_m/\nu}$ [18] with $\nu = (4\alpha)^{-1}$ except for an extra factor of $\prod_{n\neq m}\rho_{nm}^{2\alpha q_n}$. Figure 1(b) shows that $|\prod_{m(\neq n)}\rho_{nm}|$ varies only little with n for $N = 100$, and so the correspondence between the proposed wave functions and the Laughlin states is again approximately valid [note that the phase of $\prod_{m(\neq n)}\rho_{nm}^{2\alpha q_n}$ can be absorbed in χ_{n,s_n}]. Here, and in the following, we choose the distribution of the spins on the sphere by minimizing $\sum_{i<j} d_{ij}^{-2}$ numerically.

Hamiltonian.—For $\alpha = 1/2$, the vertex operators can be regarded as representations of spin 1/2 fields in the $SU(2)_1$ Wess-Zumino-Witten (WZW) model. Using properties of null fields in this model and the Ward identity, we derive [16] a set of positive semidefinite and Hermitian operators

$$H_i = \frac{1}{2}\sum_{j(\neq i)} |w_{ij}|^2 - \frac{2i}{3}\sum_{j\neq k(\neq i)} \bar{w}_{ij} w_{ik} \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k) + \frac{2}{3}\sum_{j(\neq i)} |w_{ij}|^2 \mathbf{S}_i \cdot \mathbf{S}_j + \frac{2}{3}\sum_{j\neq k(\neq i)} \bar{w}_{ij} w_{ik} \mathbf{S}_j \cdot \mathbf{S}_k, \quad (4)$$

$i = 1, \ldots, N$, which annihilate the state (2). In (4), $w_{ij} = g(z_i)/(z_i - z_j) + h(z_i)$, where g and h are arbitrary

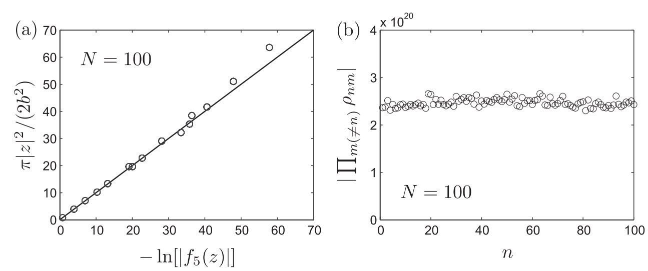

FIG. 1. (a) Comparison between $-\ln[|f_5(z)|]$ and $-\ln[|f_\infty(z)|] = \pi|z|^2/(2b^2)$ + constant. The points almost fall on a straight line with unit slope (solid line). (b) Plot of $|\prod_{m(\neq n)}\rho_{nm}|$ as a function of n for 100 spins on a sphere.

functions of z_i, and $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ is the spin operator of the ith spin. It follows that (2) is the ground state of H_i, and thus also of $H = \sum_i H_i/4 + (N+1)\sum_{i,j}\mathbf{S}_i \cdot \mathbf{S}_j/6$.

We note that H reduces to the HS Hamiltonian [11,12] when $z_n = \exp(2\pi i n/N)$ and $w_{ij} = 2z_i/(z_i - z_j) - 1$, and the construction is hence a generalization of the HS model to 2D and nonuniform distributions of the spins. In [16], we show that if the ground state is degenerate, then the additional ground states will not satisfy the Knizhnik-Zamolodchikov (KZ) equation [19]. The KZ equation is derived from the Sugawara construction that builds the Virasoro generators in terms of the Kac-Moody currents. So a degeneracy would indicate theories obeying the Kac–Moody but not the Virasoro algebra. It would be surprising if such theories exist, and we therefore expect the ground state to be unique. Exact diagonalization of H for small systems [see Fig. 2(a) for examples] also suggests uniqueness.

On the sphere, we can obtain a Hamiltonian, which is invariant under $SU(2)$ transformations of (u, v) by choosing $H = \sum_i |u_i|^{-2}[H_i^{(1)} + H_i^{(2)}]$, where $H_i^{(1)}$ $[H_i^{(2)}]$ is (4) with $w_{ij} = 1/(z_i - z_j)$ $[w_{ij} = z_i/(z_i - z_j)]$. Finally, we note that a Hamiltonian for the case $\alpha = 1/4$ can be constructed by summing single-particle Hamiltonians, each of which is the identity minus the sum of the projections onto the orthonormalized single-particle states.

Properties.—To further demonstrate the closeness between (2) and the Laughlin states, we compute various properties of (2) in the following. We note that all the numerical results presented below except those related to entanglement spectra are independent of χ_{p,s_p}.

Spin-spin correlation function: Since the systems we consider are too big for exact numerical computations, we use the Metropolis Monte Carlo algorithm to compute the spin-spin correlation function

$$\langle S_i^z S_j^z \rangle = \frac{\sum_{s_1,\ldots,s_N} s_i s_j |\psi_{s_1,\ldots,s_N}(z_1,\ldots,z_N)|^2}{4\sum_{s_1,\ldots,s_N} |\psi_{s_1,\ldots,s_N}(z_1,\ldots,z_N)|^2}. \quad (5)$$

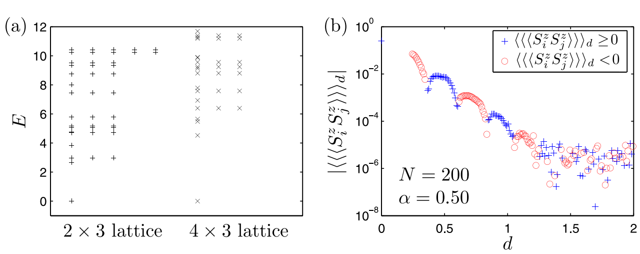

FIG. 2 (color online). (a) Low-lying part of the energy spectrum for a 2×3 (+) and a 4×3 (×) square lattice centered at the origin. (The horizontal axis shows the degeneracy.) (b) Averaged spin–spin correlation function for 200 spins on the sphere as a function of the distance between the spins. The error bars are of order a few times 10^{-5}, and so the results are only converged for $d \lesssim 1.2$. The symbols encode the sign of the correlations (plus for positive and circle for negative).

For $\alpha = 1/2$, the state is $SU(2)$ invariant and so $\langle S_i^a S_j^b \rangle = \delta_{ab}\langle S_i^z S_j^z \rangle$, $a, b = x, y, z$. Figure 2(b) shows the average of the spin–spin correlation function for 200 spins on the sphere and $\alpha = 1/2$ as a function of the distance between the spins. For a given d, the average is taken over all spin pairs for which the distance d_{ij} between the spins falls within the interval $[d - \epsilon, d + \epsilon]$, where $\epsilon = 0.005$. As for the Laughlin state with $\nu = 1/2$, we observe antiferromagnetic oscillations and exponential decay of the correlations.

The dependence of the spin-spin correlation function on α is investigated in Fig. 3. Except for α close to 0.25, we find that the correlator decays approximately as $\exp(-d/\xi)$, where d is the distance between the spins and ξ is the correlation length. Numerical estimates of ξ^{-1} are shown in Fig. 3(a). We also find that antiferromagnetic oscillations occur above $\alpha = 0.25$, but not below. This observation is consistent with the lack of oscillations in the correlation function for the $\nu = 1$ Laughlin state and the presence of oscillations for filling factors below unity. It is also consistent with the conjecture that the transition occurs precisely at $\nu = 1$ [20,21]. The transition is illustrated in plots (b)–(d). Finally, an analytical expression for the correlation function of the continuous $\nu = 1$ Laughlin state on the sphere has been found in [22], and the figure shows good agreement on intermediate length scales.

EE: The possibility of having quasiparticles with fractional statistics is a very important aspect of the Laughlin states, and it is therefore very relevant to check whether (2) also has nontrivial topological properties. Here, we study the EE, since this allows us to extract the total quantum dimension D [23]. More precisely, if we divide a system

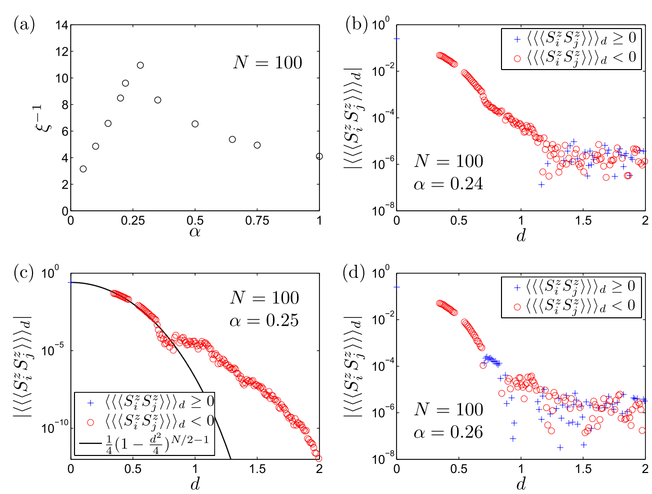

FIG. 3 (color online). (a) Inverse correlation length as a function of α for 100 spins on a sphere. (b–d) Averaged spin–spin correlation function as a function of the distance between the spins for (b) $\alpha = 0.24$, (c) $\alpha = 0.25$, and (d) $\alpha = 0.26$. The error bars in (b) and (d) are of order 10^{-5} for all points, while the results in (c) are exact. The solid curve in (c) is minus the expression for the correlation function of the continuous $\nu = 1$ Laughlin state on the sphere found in [22].

into two subsystems A and B and the system is gapped, the EE has the form $aP - \gamma + \ldots$ [24], where a is a constant, P is (proportional to) the length of the boundary between A and B, $-\gamma$ is called the topological EE and fulfils $\gamma = \ln(D)$ [24,25], and the ellipsis stands for terms that vanish for $P \to \infty$. For Abelian systems, D^2 is the number of different quasiparticles one can get by fusing the fundamental quasiparticles in the system, and so $D = \sqrt{1/\nu}$ for the Laughlin states with $1/\nu \in \mathbb{N}$ [23].

The linear increase of the EE with P is confirmed in Fig. 4(a). We use here the Rényi entropy $S_L^{(2)} = -\ln[\mathrm{Tr}(\rho_A^2)]$, where ρ_A is the reduced density operator of the spins in region A, because this quantity can be computed efficiently using Monte Carlo methods [26,27]. The boundary between A and B is assumed to be the circle $\theta = \theta_L$, where θ_L is defined such that the area on the sphere with $\theta < \theta_L$ relative to the complete area of the sphere equals L/N. The length of the boundary is then proportional to $P \equiv (L/N)^{1/2}(1 - L/N)^{1/2}$, and the steps in the EE appear due to the discreteness of the positions of the spins. For the Laughlin states, there are more ways in which the system can naturally be divided into two parts, and previous studies [28,29] have computed EEs for both orbital partitioning, in which region A involves a subset of angular momentum eigenstates, and particle partitioning, in which a particular subset of the spins comprises part A of the system. We note that such complications do not appear here, since the spins are fixed at specific positions.

The topological EE cannot be read off reliably from Fig. 4(a) because small errors in the linear term due to irregularities in the boundary easily become dominant. Instead, we use the method proposed in [24], which eliminates the linear term by considering a linear combination of EEs of suitably chosen regions. The results in [24] are for the von Neumann entropy, but it has been shown in [30] that $\gamma = \ln(D)$ also holds for $S_L^{(2)}$ when region A is a topologically trivial region. For $\alpha = 1/4$, we find $\gamma = 0$, and for $\alpha = 1/2$, we get $-\gamma = -0.341 \pm 0.057$ from a Monte Carlo simulation involving 160 spins on a sphere. Both results are consistent with the expected values [0 and $-\ln(2)/2 \approx -0.347$, respectively] and with the results obtained for lattice models in [31].

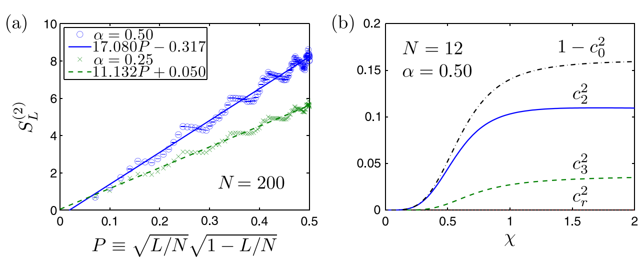

FIG. 4 (color online). (a) Rényi entropy $S_L^{(2)} = -\ln[\mathrm{Tr}(\rho_A^2)]$ for 200 spins on a sphere for $\alpha = 1/2$ (blue circles) and $\alpha = 1/4$ (green crosses) when region A is chosen to be the L spins that are closest to the North Pole. The solid and dashed lines are linear fits. (b) The coefficients in (6) for $N = 12$.

Entanglement spectrum: It has turned out (typically from numerical studies) that the low-lying part of the entanglement spectrum, defined as the eigenvalues of $-\ln(\rho_A)$, is often related to some theory on the boundary of A [32]. Utilizing the particular structure of the wave functions (2), we can here derive such a connection analytically for the case of $\alpha = 1/2$ and N spins distributed uniformly on two rings in the complex plane; i.e., the spins are at the positions $\exp[2\pi(2in \pm \chi)/N]$, where $n = 1, 2, \ldots, N/2$. Specifically, we prove in the Supplemental Material [16] that ρ_A (the reduced density operator of the inner ring) is invariant under Yangian transformations, which means that $-\ln(\rho_A)$ is a linear combination of the invariants in the HS model. More precisely, we can write $-\ln(\rho_A)$ as a linear combination of the identity H_0, the two-body operator $H_2 = 2\sum_{n \neq m} \mathbf{S}_n \cdot \mathbf{S}_m z_n z_m / z_{nm}^2$, $z_{nm} \equiv z_n - z_m$, the three-body operator $H_3 = -i\sum_{n \neq m \neq p} \mathbf{S}_n \cdot (\mathbf{S}_m \times \mathbf{S}_p) z_n z_m z_p / (z_{nm} z_{mp} z_{pn})$, and operators with higher body interactions, which we write as H_r [33]. Considering these operators as normalized vectors $|H_i\rangle$ with inner product $\langle H_i | H_j \rangle = \mathrm{Tr}(H_i H_j) / [\mathrm{Tr}(H_i^2)(H_j^2)]^{1/2}$, we can write

$$|-\ln(\rho_A)\rangle = c_0|H_0\rangle + c_2|H_2\rangle + c_3|H_3\rangle + c_r|H_r\rangle. \quad (6)$$

The coefficients are given for $N = 12$ in Fig. 4(b). We note that the above results do not generalize to the case of more than two rings.

Conclusion.—We have proposed a set of spin system wave functions that is in many respects analogous to the Laughlin states. The proposed mapping between spin states and Laughlin states builds on CFT and demonstrates the usefulness of CFT as a tool to gain insight into many-body systems. We believe that similar mappings can be found also for other quantum Hall states. In particular, one can obtain spin analogies of the Moore–Read state by generalizing the higher level 1D spin models proposed in [10] to 2D, which can be done straightforwardly.

The analogy between Laughlin states and spin states can be carried even further since the method proposed in [9] to incorporate quasiholes by introducing additional conformal operators can also be used for the state (2). As for the Laughlin states, one can interpret the square of the norm of the spin wave function as a particular charge distribution, and we expect this distribution to screen the quasiholes. It follows immediately from the construction that the analytic continuation properties of the wave functions are the same as for the Laughlin states with quasiholes. It would be interesting to investigate these ideas further.

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[16] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.108.257206 for further details on the singlet property of the wave function for $\alpha = 1/2$, the proof that the modified and original Laughlin states are the same on an infinite square lattice, the derivation of the Hamiltonian (4), the argument that there is only one state that is both a ground state and a solution to the KZ equation, and the proof that ρ_A is invariant under Yangian transformations.
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