Symmetry-Protected Phases for Measurement-Based Quantum Computation

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Ground states of spin lattices can serve as a resource for measurement-based quantum computation. Ideally, the ability to perform quantum gates via measurements on such states would be insensitive to small variations in the Hamiltonian. Here, we describe a class of symmetry-protected topological orders in one-dimensional systems, any one of which ensures the perfect operation of the identity gate. As a result, measurement-based quantum gates can be a robust property of an entire phase in a quantum spin lattice, when protected by an appropriate symmetry.

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Quantum computation exploits quantum entanglement to achieve computational speedups. However, creating entanglement between particles in a sufficiently controlled way to allow for quantum computation has proved a major technical challenge. One potential approach is measurement-based quantum computation (MBQC) [1,2], where universal quantum computation is achieved by means of nonentangling operations (namely, single-particle measurements) on an already entangled resource state. The resource state need not be prepared coherently; instead, one could imagine constructing interactions between neighboring spins on a lattice, governed by a gapped Hamiltonian whose ground state is a universal resource state for MBQC [3–5]. For this approach to be robust, the capability of ground states to serve as a resource for MBQC would have be insensitive to small variations in the Hamiltonian, like a form of quantum order [3].

In this Letter, we draw an explicit connection between MBQC and a type of quantum order called symmetry-protected topological order (SPTO) [6–8]. Specifically, we will describe a class of quantum phases in which the perfect operation of the identity gate in MBQC can be derived directly from the presence of SPTO; consequently, this perfect operation is a robust property which is maintained throughout the entire phase. Our results will be expressed in the context of one-dimensional (1D) systems. Such systems are not expected to allow for universal MBQC, but the ground states of certain 1D spin chains can be used as quantum computational wires [9], meaning, loosely, that through single-particle measurements one can propagate a logical qubit down the chain while applying single-qubit unitaries. Later, we will also explain how our results can be applied to higher-dimensional systems (which can allow for universal MBQC) by considering them as "quasi-1D".

A well-known example of a one-dimensional system whose ground state can serve as a quantum computational wire is the Affleck-Kennedy-Lieb-Tasaki (AKLT) antiferromagnetic spin-1 chain [10,11]. This system lies in a quantum phase, called the Haldane phase, characterized by SPTO and protected by a $Z_2 \times Z_2$ rotation symmetry [12,13], so that no symmetry-respecting path of local Hamiltonians can interpolate between the Haldane phase and a product state without crossing a phase transition. The perfect operation of the identity gate throughout the Haldane phase has been noticed before in various guises [14,15], as well as the strictly weaker condition of diverging localizable entanglement length [16]. It should be emphasized that in MBQC, repeated application of the identity gate corresponds to the propagation of a logical state arbitrarily far down the chain without error. Thus, the identity gate is not a null operation in this context, and its perfect operation is a striking and nontrivial property of the Haldane phase.

Our purpose in this Letter will be to show explicitly how the perfect operation of the identity gate arises as a direct manifestation of SPTO. As a result, we can apply our technique more generally to a whole class of quantum phases characterized by SPTO, including phases containing the 1D cluster state, qudit cluster states [17], and cluster states in higher dimensions. In addition, we show that gates other than the identity are not expected to exhibit similar robustness, explaining the numerical observations in Ref. [15].

Symmetry protection of the identity gate in correlation space.—The connection between SPTO and MBQC will be expressed through the correlation-space picture of Ref. [18], which is a particularly natural way to understand the operation of gates in 1D resource states. This picture assumes a resource state $|\Psi\rangle$ that can be represented as a matrix-product state (MPS),

$$|\Psi\rangle = \sum_{k_1,\ldots,k_N} \langle R|A[k_N]A[k_{N-1}]\cdots A[k_1]|L\rangle \times |k_1,\ldots,k_N\rangle, \tag{1}$$

where each $A[k_j]$, $k_j = 1,\ldots,d$ is a linear operator acting on a D-dimensional vector space (known as the correlation space), $|L\rangle$ and $\langle R|$ are states in correlation space, and d is the dimension of the Hilbert space of each spin. Here we are assuming translational invariance, for notational simplicity only. When a projective measurement is performed on the first spin, with outcome $|\psi\rangle$, the effect is to remove the first spin from the chain and induce an evolution $|L\rangle \rightarrow A[\psi]|L\rangle$ on the correlation system, where we use the notation $A[\psi] = \sum_k A[k]\langle\psi|k\rangle$.

As an introduction to our result, we will first state it for the special case of the Haldane phase. One system within this phase is the spin-1 AKLT chain, for which the ground state has an exact MPS representation of the form Eq. (1), with $D = 2$. Expressed in the basis $\{|x\rangle, |y\rangle, |z\rangle\}$, where $|\alpha\rangle$ is the zero eigenstate of the spin-1 operator S_α for $\alpha = x$, y, z, we have $A^{\mathrm{AKLT}}[\alpha] = \sigma_\alpha$, where σ_α are the Pauli spin operators. Thus, the AKLT state has the particular property that there exists a basis, namely the $\{|x\rangle, |y\rangle, |z\rangle\}$ basis, such that measurements in this basis induce an identity evolution (up to Pauli by-products) on the correlation system. Additionally, by measuring in a basis corresponding to a rotated set of axes, it is possible to execute any single-qubit rotation in correlation space (up to Pauli by-products) [11]. Therefore, the AKLT state can be said to act as a quantum computational wire.

We will now extend our correlation-space analysis beyond the AKLT chain to other ground states within the Haldane phase. We confine our discussion to states that can be exactly represented as an MPS with a bond dimension D that is independent of the system size. Because arbitrary gapped ground states can be approximated by MPS [19], we expect that our discussion will apply also to arbitrary systems in the Haldane phase.

The Haldane phase containing the AKLT chain is protected by the $Z_2 \times Z_2$ symmetry generated by the π rotations about three orthogonal axes. The action of this symmetry on a spin-1 chain can be written as a tensor product $[u(g)]^{\otimes N}$, where N is the number of spins and $u(g)$ is the appropriate single-spin rotation operator for each group element g in the symmetry group $G = Z_2 \times Z_2$. We therefore refer to it as an on-site symmetry.

In general, the invariance of a ground state under such an on-site symmetry leads to symmetry constraints on the MPS tensor $A[\cdot]$ used to construct the state's MPS representation [7,8,20,21]; we will exploit these constraints to prove our result. Specifically, under an injectivity assumption which we expect to be satisfied in a gapped phase, we have [7,8,20]

$$V(g)^\dagger A[|\psi\rangle]V(g) = \beta(g)A[u(g)^\dagger|\psi\rangle], \tag{2}$$

where $V(g)$ is some projective representation of G acting on the correlation system, and $\beta(g)$ is a one-dimensional linear representation of G. Now, in general $V(g)$ can be decomposed as a tensor sum of irreducible projective representations as $V(g) = \bigoplus_J V_J(g) \otimes \mathbb{I}_{m_J}$, where m_J is the multiplicity of the irrep J in V. For any ground state in the Haldane phase, it is a consequence of Lemma 2 below that only one irrep $\tilde{V}(g)$ (of dimension 2) appears in this decomposition, so that

$$V(g) = \tilde{V}(g) \otimes \mathbb{I}_{\mathrm{junk}}. \tag{3}$$

That is, we have a tensor product decomposition of the correlation system into a protected subsystem [on which $V(g)$ acts irreducibly as $\tilde{V}(g)$] and a junk subsystem (on which $V(g)$ acts trivially). The states $|x\rangle$, $|y\rangle$, $|z\rangle$ are simultaneous eigenstates of all the elements $u(g)$. By an argument involving Schur's Lemma (given in greater generality in Theorem 1), it follows that the tensor A appearing in the MPS representation of the ground state must take the form

$$A[\alpha] = \sigma_\alpha \otimes A_{\mathrm{junk}}[\alpha], \qquad \alpha = x, y, z, \tag{4}$$

for some set of operators $A_{\mathrm{junk}}[\alpha]$ acting on the junk subsystem. Recall that $A[\alpha]$ is the evolution induced on the correlation system when a projective measurement results in the outcome $|\alpha\rangle$. Thus, Eq. (4) shows that the ability to induce an identity evolution in the protected subsystem (up to Pauli by-products, dependent on the measurement outcome but independent of the resource state) by measuring in the $\{|x\rangle, |y\rangle, |z\rangle\}$ basis is dictated by the symmetry properties of the MPS tensor; it is a property not just of the AKLT state, but rather of the entire Haldane phase.

Another state which can serve as a quantum computational wire is the 1D cluster state, which is the ground state on a row of qubits of the local Hamiltonian $H = -\sum_i Z_{i-1}X_iZ_{i+1}$. Like the AKLT state, the cluster state has an exact MPS representation, and it lies within a symmetry-protected phase with respect to a $Z_2 \times Z_2$ symmetry [22], in this case generated by $\prod_{i\,\mathrm{even}} X_i$ and $\prod_{i\,\mathrm{odd}} X_i$. We can treat this symmetry as on-site provided that we group pairs of qubits into sites. The simultaneous eigenbasis of the on-site symmetry representation is then $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ (we emphasize that this is a product basis, so that blocking sites does not change the single-qubit nature of the measurements). Identical to the AKLT case above, we again find that the ability to perform the identity gate by measuring in the appropriate basis is maintained throughout the phase. Similar results hold for the generalization of the cluster state to d-dimensional particles [17], for which the relevant symmetry group is $Z_d \times Z_d$.

General statement of the result.—We will now give the statement and proof of our result in a general setting. We consider a ground state that is invariant under an on-site symmetry $[u(g)]^{\otimes N}$, where $u(g)$ is a representation of some symmetry group G. We assume the ground state has an MPS representation satisfying the symmetry condition (2), and we absorb $\beta(g)$ into $u(g)$ so that $\beta(g) = 1$. A projective representation $V(g)$ is characterized by its factor system ω, such that

$$V(g)V(h) = \omega(g, h)V(gh). \quad (5)$$

An equivalence class of factor systems related by rephasing of the operators $V(g)$ is called a cohomology class, and we denote the cohomology class containing a given factor system ω as $[\omega]$. It was argued in Refs. [7,8] that each cohomology class of G corresponds to a distinct symmetry-protected phase. For example, in the case of the MPS $A^{\text{AKLT}}[\alpha] = \sigma_\alpha$ for the AKLT state, where $G = Z_2 \times Z_2 = \{1, x, y, z\}$, it can be verified that Eq. (2) is satisfied with the Pauli projective representation $V(1) = I$ and $V(\alpha) = \sigma_\alpha$ for $\alpha = x, y, z$. This corresponds to a nontrivial cohomology class [not containing the trivial factor system $\omega(g, h) = 1$], so that the AKLT chain lies in a nontrivial symmetry-protected phase.

We now relate the symmetry condition (2), which holds throughout the entire symmetry-protected phase, to the operation of gates in the correlation-space picture. We consider the case where the symmetry group G is a finite Abelian group. For simplicity, we will focus on the case where the cohomology class $[\omega]$ characterizing the symmetry-protected phase is of a particular type. (An analogous result holds for all nontrivial cohomology classes, but the structure of correlation space is more involved in that case.) In particular, we consider the case where the factor systems contained in $[\omega]$ are maximally noncommutative, meaning that the subgroup $G(\omega) = \{g \in G | \omega(g, h) = \omega(h, g) \forall\ h \in G\}$ is trivial. (Note, this condition does not depend on the choice of the representative ω.) Under these conditions, our main result can be stated as follows:

Theorem 1.—Consider a symmetry-protected phase characterized by a finite Abelian symmetry group and a maximally noncommutative cohomology class $[\omega]$. Then for any MPS in this phase, there exists a decomposition of the correlation system into protected and junk subsystems and a site basis $\{|i\rangle\}$, such that measuring in the basis $\{|i\rangle\}$ leads to an identity gate evolution on the protected subsystem up to an outcome-dependent by-product B_i. That is to say, the MPS tensor A has the decomposition

$$A[i] = B_i \otimes A_{\text{junk}}[i]. \quad (6)$$

The by-product operators B_i are unitary and are elements of a finite group. Furthermore, they are the same for all possible MPS in the symmetry-protected phase.

For example, the factor system for the Pauli projective representation of $Z_2 \times Z_2$ is maximally noncommutative, and Eq. (4) is a special case of Eq. (6).

Proof of Theorem 1.—We will make use of the following consequences of maximal noncommutativity of a factor system:

Lemma 1.—Let ω be a maximally noncommutative factor system of a finite Abelian group G. For every linear character χ of G, there exists an element $h_\chi \in G$ such that, for any projective representation $V(g)$ with factor system ω,

$$V(h_\chi)V(g) = \chi(g)V(g)V(h_\chi). \quad (7)$$

Proof.—We define a homomorphism φ: $G \to G^*$, where G^* is the group of linear characters of G, according to $[\varphi(h)](g) = \omega(h, g)\omega(g, h)^{-1}$. (That $\varphi(g) \in G^*$ for all g and φ is a homomorphism follows from the associativity condition satisfied by ω; e.g., see Lemma 7.1 in Ref. [23]). Because the kernel of φ is $G(\omega)$, which is trivial by assumption, and $|G| = |G^*|$ for finite Abelian groups, it follows that φ is invertible. We then set $h_\chi = \varphi^{-1}(\chi)$. It can be checked that this satisfies Eq. (7). □

Lemma 2.— For each maximally noncommutative factor system ω of a finite Abelian group G, there exists a unique (up to unitary equivalence) irreducible projective representation $\tilde{V}(g)$ with factor system ω. The dimension of this irreducible representation is $\sqrt{|G|}$.

Proof.—See Refs. [24,25]. □

For an MPS tensor A satisfying the symmetry condition (2), Lemma 2 implies that there exists a tensor product decomposition of the correlation system into a protected and a junk subsystem such that $V(g)$ acts within the protected subsystem as $\tilde{V}(g)$ as in Eq. (3).

Now we can prove Theorem 1. We choose the measurement basis $\{|i\rangle\}$ to be the simultaneous eigenbasis of the elements $u(g)$, such that $u(g)|i\rangle = \chi_i(g)|i\rangle$, where each χ_i is a linear representation of G. Expressed in the basis $\{|i\rangle\}$, Eq. (2) then becomes

$$V(g)^\dagger A[i]V(g) = \chi_i(g)A[i]. \quad (8)$$

Making use of Eq. (7), we find that

$$V(g)\{V(h_{\chi_i})^\dagger A[i]\} = \{V(h_{\chi_i})^\dagger A[i]\}V(g). \quad (9)$$

We can now conclude by Schur's Lemma that

$$A[i] = \tilde{V}(h_{\chi_i}) \otimes A_{\text{junk}}[i] \quad (10)$$

for some operators $A_{\text{junk}}[i]$. Therefore, Theorem 1 holds with $B_i = \tilde{V}(h_{\chi_i})$. □

Nontrivial gates.—In Theorem 1, we have proven that the identity gate, which involves measuring in the simultaneous eigenbasis of the operators $u(g)$, is symmetry-protected. We will now see that nontrivial gates (i.e., those involving measurement in a different basis) are not symmetry protected.

For example, let us consider a measurement that on the exact AKLT state would correspond to a rotation by an angle 2θ about the z axis (up to Pauli by-products). One of the possible measurement outcomes is $|\theta\rangle \equiv \cos\theta|x\rangle + \sin\theta|y\rangle$. Then from the decomposition (4) of the MPS tensor A for a generic state in the Haldane phase, we find that

$$A[\theta] = (\cos\theta)\sigma_x \otimes A_{\text{junk}}[x] + (\sin\theta)\sigma_y \otimes A_{\text{junk}}[y]. \quad (11)$$

If $A_{\text{junk}}[x] = A_{\text{junk}}[y]$ (as for the exact AKLT state), then this implies

$$A[\theta] = [(\cos\theta)\sigma_x + (\sin\theta)\sigma_y] \otimes A_{\text{junk}}[x], \tag{12}$$

and the evolution on the protected subsystem is the same as it would be for the exact AKLT state. However, there is no symmetry constraint that guarantees $A_{\text{junk}}[x] = A_{\text{junk}}[y]$ (because any choice whatsoever for A_{junk} in Eq. (4) gives rise to an MPS satisfying the symmetry constraints). Therefore, the evolution induced by measurements in this basis is not fixed by the symmetry; similar arguments apply to all nontrivial gates.

The preceding discussion of nontrivial gates applies to systems with only the $Z_2 \times Z_2$ rotation symmetry, and larger symmetry groups will lead to stronger constraints on the MPS tensor. In particular, one might expect that for the AKLT state, imposing the full SO(3) rotation symmetry would lead to all gates being protected, because *all* gates are achieved by measuring in the basis $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ for some rotated orthogonal set of axes x', y', z'. This would indeed be true if only the spin-1/2 projective representation $V_{1/2}(g)$ of SO(3) appeared in the irrep decomposition of $V(g)$, so that $V(g) = V_{1/2}(g) \otimes \mathbb{I}$. However, all the half-integer spin representations of SO(3) have the same cohomology class, so this will not hold in general. Indeed, the numerical results of Ref. [15] show reduced performance of nontrivial gates. This should be contrasted with the protocol of Ref. [26], where a logical qubit is encoded into an explicitly spin-1/2 edge mode and particles are adiabatically decoupled from the chain before being measured. In that case it was found that all gates operate perfectly throughout the Haldane phase so long as the full rotational symmetry is maintained.

Initialization and readout.—Apart from performing unitary gates in correlation space, the other essential ingredient for MBQC is the ability to initialize and read out the state of the correlation system. It is easily verified (in the same way as for nontrivial gates) that the usual procedures for doing this in the cluster or AKLT states are not symmetry-protected. However, a symmetry-protected readout *can* be achieved throughout the Haldane phase by terminating a finite chain of spin-1 particles with a spin-1/2, as in Ref. [15].

Higher-dimensional systems.—The notion of symmetry-protected topological order has recently been extended to higher-dimensional systems [27,28], and we speculate that our results could be generalized in this context. However, if we consider a "quasi-1D" system whose extent in all but one dimension is finite (but could be set arbitrarily large), then the results of this Letter can be applied directly.

For example, a 2D cluster model of extent $2N$ in the vertical direction (with periodic boundary conditions in that direction) is in a nontrivial symmetry-protected phase with respect to the $(Z_2 \times Z_2)^{\times N}$ symmetry depicted in Figure 1. This symmetry is represented in correlation space by a tensor product of N copies of the Pauli representation; this is a maximally noncommutative projective representation of the symmetry group. By Lemma 2, the protected subsystem has dimension 2^N. Therefore, throughout the symmetry-protected phase there is a capacity for N qubits to be propagated in the horizontal direction by measuring each "site" (here a pair of adjacent columns) in a simultaneous eigenbasis of the symmetry. For the particular representation of $(Z_2 \times Z_2)^{\times N}$ depicted in Figure 1, it is straightforward to show that there exists such an eigenbasis which is also a product basis over the qubits making up the site, so that this propagation can be achieved by single-qubit measurements.

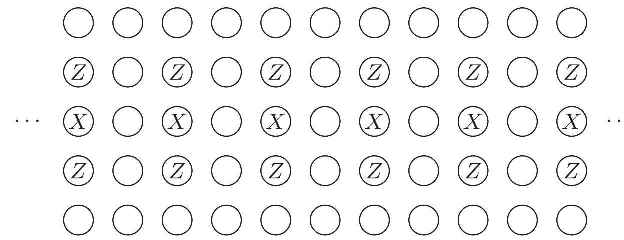

FIG. 1. One generator of the $(Z_2 \times Z_2)^{\times N}$ symmetry in the 2D cluster state. The other generators can be obtained from this one by a displacement of 1 horizontally and/or an even number vertically. The circles represent qubits in the 2D square lattice.

Conclusion.—In summary, we have identified a class of symmetry-protected topological orders, each of which ensures the perfect operation of the identity gate in MBQC throughout an entire symmetry-protected phase. Such connections between MBQC and quantum order can be expected to lead to a greater understanding of the potential for single-particle measurements on ground states of quantum spin systems to be a robust form of quantum computation.

By contrast, we have shown that the perfect operation of nontrivial gates is a property only of specific systems within such a phase, contrary to some previous hopes [3]. However, we have not given a complete characterization of the operation of nontrivial gates away from these points, and it is possible that their performance could be made arbitrarily good by a suitable choice of adaptive measurement protocol, as in Ref. [15].

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