Increasing Entanglement Monotones by Separable Operations

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Quantum entanglement is fundamentally related to the operational setting of local quantum operations and classical communication (LOCC). A more general class of operations known as separable operations (SEP) is often employed to approximate LOCC, but the exact difference between LOCC and SEP is unknown. In this letter, we compare the two classes in performing particular tripartite to bipartite entanglement conversions and report a gap as large as 12.5% between SEP and LOCC, which is the first known appreciable gap between the classes. Our results rely on constructing a computable entanglement monotone with a clear operational meaning that, unlike all other such monotones previously studied, is not monotonic under SEP. Finally, we prove the curious fact that convergent sequences of LOCC protocols need not be LOCC feasible in the limit.

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In the ''distant lab'' setting of quantum information processing, entanglement is shared among spatially separated parties, and its manipulation is implemented through local quantum operations coordinated by classical communication (LOCC). A general LOCC operation consists of each party taking turns in measuring his or her part of the system and then broadcasting this result, which may affect the choice of future measurements performed by other parties. Understanding the possibilities and limitations of LOCC processing is of great practical importance since many quantum information tasks such as teleportation [1], entanglement distillation [2], one-way quantum computation [3], and data hiding [4,5] are based on the LOCC paradigm.

From a deeper perspective, LOCC operations play a crucial role in entanglement theory, as it is a fundamental rule that entanglement cannot increase under LOCC processing [6–9]. Making this statement more precise involves defining measures for entanglement, and when doing so it is important to distinguish between the so-called finite and asymptotic ''regimes'' [10,11]. In the finite case, the object of interest is a *single state* ρ (or finite copies $\rho^{\otimes n}$), and we are concerned with the entanglement possessed by ρ (or $\rho^{\otimes n}$). In contrast, for the asymptotic setting it is more appropriate to treat the system as a *source* of some quantum state ρ, and the object of interest is the many-copy limit $\lim_{n\to\infty}\rho^{\otimes n}$. Consequently, when treating asymptotic entanglement, the focus shifts from particular states *per se* to their close approximations in the many-copy limit. The finite and asymptotic cases are each of independent interest, and their relevance depends on the particular physical scenario. Throughout this Letter, we will focus on finite-copy entanglement.

In the axiomatic approach to entanglement measures, necessary conditions are specified for what any entanglement measure must satisfy. While there is no universal agreement on these axioms, in the finite-copy regime, the essential property for some non-negative function μ to qualify as an entanglement measure is monotonicity under LOCC: for an arbitrary state ρ and LOCC transformation converting ρ into ρ_λ with probability p_λ, the inequality $\mu(\rho) \geq \sum_\lambda p_\lambda \mu(\rho_\lambda)$ necessarily holds [9,10,12–14]. Any function that satisfies this condition is called an *entanglement monotone*.

The underlying effect of the axiomatic approach in the non-asymptotic setting is that we *define* entanglement as whatever cannot be increased by LOCC [7,13,15]. Therefore, an average increase of any entanglement monotone μ under a non-LOCC process (i.e., a violation of the previous inequality) can rightfully be interpreted as an increase in entanglement. In this Letter, we will construct an entanglement monotone and provide an example of such an increase when the non-LOCC process is a so-called separable operation (defined below).

Despite its fairly intuitive physical description, general LOCC operations are quite challenging to analyze [11,16,17]. This difficulty can be somewhat alleviated by considering the more general class of separable operations (SEP), which for N parties consists of all completely positive maps that allow for a representation of the form $\mathcal{E}(\cdot) = \sum_\lambda A_\lambda(\cdot)A_\lambda^\dagger$, where $A_\lambda = M_{1,\lambda}\otimes M_{2,\lambda}\otimes\ldots\otimes M_{N,\lambda}$. Using SEP to study LOCC has been useful for computing bounds on entanglement distillation [12,18,19], deciding distinguishability of states [20,21], and proving the bound nature of PPT entanglement [22].

While LOCC $\subset$ SEP and the two share many similarities, it is well known that LOCC $\neq$ SEP [23,24]. A dramatic example of this latter fact is the phenomenon of ''non-locality without entanglement,'' a term originally used to describe sets of orthogonal product states indistinguishable by LOCC. Unfortunately, the exact difference between LOCC and SEP is not well understood, and

Ref. [23] could only quantify a tiny gap between the two of order 10^{-6}. Later in Ref. [25], another gap between SEP and LOCC was reported in two qubits; however, this was only on the order of 0.8%. A key contribution of this Letter is finding, for the first time, a *computable* and analytic LOCC monotonic function with a clear *operational* meaning that, in fact, does increase under SEP. This allows us to demonstrate an *appreciable* 12.5% operational gap between SEP and LOCC.

A final consequence of our findings is that convergent sequences of LOCC protocols need not be implementable by LOCC. What this means is that there exists a sequence of LOCC maps $\mathcal{L}_1, \mathcal{L}_2, \ldots$ converging to some map $\bar{\mathcal{L}}$ that cannot be carried out through LOCC processing. This is rather surprising since in contrast, global quantum operations, separable operations, and even LOCC operations restricted to one-way communication are all closed under convergent sequences of maps [26].

To obtain our results, we will focus on the task of random-pair EPR distillation from a three-qubit W-class state [28,29]. A W-class state is any state obtainable from $|W\rangle = \sqrt{\frac{1}{3}}(|100\rangle + |010\rangle + |001\rangle)$ by stochastic LOCC [30], and up to a local unitary (LU) transformation, it takes the form $|\varphi\rangle = \sqrt{x_0}|000\rangle + \sqrt{x_A}|100\rangle + \sqrt{x_B}|010\rangle + \sqrt{x_C}|001\rangle$, where $x_0 = 1 - x_A - x_B - x_C$. In fact, the vector $\vec{x} = (x_A, x_B, x_C)$ uniquely characterizes the $|\varphi\rangle$ such that if $|\varphi\rangle = \sqrt{x_0'}|0'0'0'\rangle + \sqrt{x_A'}|1'0'0'\rangle + \sqrt{x_B'}|0'1'0'\rangle + \sqrt{x_C'}|0'0'1'\rangle$ for some other basis $\{|0'\rangle, |1'\rangle\}$, then $x_i = x_i'$ [31]. A random-pair EPR distillation of $|\varphi\rangle$ consists of the multi-outcome transformation

$$|\varphi\rangle \rightarrow \{p_{ij}, |\Psi^{(ij)}\rangle\}, \quad (1)$$

in which $|\Psi^{(ij)}\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$ is a maximally entangled state shared between parties i and j, obtained with probability p_{ij} from $|\varphi\rangle$. When the initial state is $|W\rangle$, Fortescue and Lo have devised a family of protocols that for any $\epsilon > 0$ can achieve this transformation with probability $p_{AB} + p_{AC} + p_{BC} > 1 - \epsilon$ [28].

We can better analyze the random distillation problem by examining how W-class states transform under local measurements. Up to LU transformations, a general local measurement by party k can be represented by Kraus operators $\{M_\lambda^{(k)}\}$, where

$$M_\lambda^{(k)} = \begin{pmatrix} \sqrt{a_{k,\lambda}} & b_{k,\lambda} \\ 0 & \sqrt{c_{k,\lambda}} \end{pmatrix}$$

[31]. The diagonal elements satisfy $\sum_\lambda a_\lambda = 1$ and $\sum_\lambda c_\lambda \leq 1$. For a general W-class state with vector representation $\vec{x} = (x_A, x_B, x_C)$, it is straightforward to compute that the components transform as $x_k \rightarrow x_{k,\lambda} = \frac{c_{k,\lambda}}{p_\lambda} x_k$ and $x_j \rightarrow x_{j,\lambda} = \frac{a_{k,\lambda}}{p_\lambda} x_j$ for $j \neq 0, k$, and each outcome λ. It then follows that [31]

$$x_0 \leq \sum_\lambda p_\lambda x_{\lambda,0} \qquad x_i \geq \sum_\lambda p_\lambda x_{\lambda,i} \quad \forall_{i\in\{A,B,C\}}. \quad (2)$$

Next, we introduce two continuous functions defined on the class of three-qubit W-class states. For a state $\vec{x} = (x_A, x_B, x_C)$, let $\{n_1, n_2, n_3\} = \{A, B, C\}$ such that $x_{n_1} \geq x_{n_2} \geq x_{n_3}$. Then define the functions

$$\eta(\vec{x}) := x_{n_2} + x_{n_3} - x_{n_2}x_{n_3}/x_{n_1}, \quad (3)$$

$$\kappa(\vec{x}) := x_{n_2} + x_{n_3} + \eta(\vec{x}). \quad (4)$$

Note that $\eta(s\vec{x}) = s\eta(\vec{x})$ and $\kappa(s\vec{x}) = s\kappa(\vec{x})$, where s is any overall scaling to the elements of $\vec{x}$. Also, in the two qubit case $x_0 = x_{n_3} = 0$, and so $\kappa(\vec{x})$ reduces to $2x_{n_2}$, a known bipartite entanglement monotone [32].

Theorem 1.—(A) The function η is non-increasing on average when party n_1 does not change among the pre-measurement and all the post-measurement states. (B) The function κ is always an entanglement monotone that strictly decreases on average whenever a party with maximum component value makes a measurement.

Proof.—To prove these claims, it is sufficient to consider weak measurements that have only two outcomes [33,34]. This is because any measurement can be decomposed into a sequence of two outcome measurements, which can be further decomposed into a sequence of weak binary outcome measurements; by continuity, the functions η and κ will vary as smoothly as desired during a general transformation. Thus without loss of generality, the Kraus operators for the local measurements take the form

$$M_1 = \begin{pmatrix} \sqrt{a} & b_1 \\ 0 & \sqrt{c_1} \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} \sqrt{1-a} & b_2 \\ 0 & \sqrt{c_2} \end{pmatrix},$$

with a, c_1, $c_2 \approx 1/2$. (A) First consider when party n_1 makes a measurement. Then

$$\Delta\bar{\eta} = x_{n_2}x_{n_3}/x_{n_1}(1 - a^2/c_1 - (1-a)^2/c_2). \quad (5)$$

Noting that $c_2 \leq 1 - c_1$, we expand Eq. (5) about the point $(a, c_1) = (1/2, 1/2)$ to obtain

$$\Delta\bar{\eta} \leq -4(a-c)^2 + O((c-1/2)^3, (a-1/2)^3) < 0. \quad (6)$$

On the other hand, if either party n_2 or n_3 measures, we can use the observation that $\eta(\vec{x}) = x_{n_2}(1 - x_{n_3}/x_{n_1}) + x_{n_3} = x_{n_3}(1 - x_{n_2}/x_{n_1}) + x_{n_2}$ along with Eq. (2) to see that η is non-increasing on average. (B) By the decomposition of a general transformation into a sequence of weak measurements, each measurement either satisfies the conditions of (A), or its pre-measurement state $\vec{y}$ satisfies $y_{n_1} = y_{n_2}$. In the first case, we know that κ is monotonic by part (A) and the fact that $x_{n_2} + x_{n_3}$ is non-increasing on average. On the other hand, when $y_{n_1} = y_{n_2}$, we have $\kappa(\vec{y}) = 1 - y_0$. The measurement on $\vec{y}$ will yield outcomes $\vec{y}_1$ and $\vec{y}_2$ for which $1 - y_{0,i} \geq \kappa(\vec{y}_i)$. Thus, the monotonic nature of $-y_0$ implies that κ is non-increasing on average. Finally, note that whenever party n_1 measures, one of the outcome branches will satisfy the conditions of (A) during part of the transformation. Therefore, inequality (6) must hold at some point, and hence κ will be strictly decreasing overall. ■

We have that $\kappa(\vec{x}) = 1$ if and only if $x_{n_1} = x_{n_2}$ and $x_0 = 0$. Thus for the random-party distillation (1) of $\vec{x}$, it follows that $p_{AB} + p_{AC} + p_{BC} \leq \kappa(\vec{x})$. However, this bound is only asymptotically tight, as the next theorem shows.

Theorem 2.—Let $\vec{x}$ satisfy $x_{n_3} > x_0 = 0$. Then for every $\epsilon > 0$ there exists an LOCC transformation that completes (1) with overall success probability at least $\kappa(\vec{x}) - \epsilon$. However, for no LOCC transformation is the success probability exactly $\kappa(\vec{x})$.

Proof.—We first construct an explicit protocol consisting of measurements satisfying $c_1 + c_2 = 1$. Party n_2 first performs the binary-outcome measurement given by $a_1 = x_{n_2}/x_{n_1}$ and $c_1 = 1$. If outcome 2 occurs, party n_2 is disentangled, and parties n_1 and n_3 optimally transform into $|\Psi^{(n_1 n_3)}\rangle$ [32]. If outcome 1 occurs, party n_3 performs a measurement given by $a_1 = x_{n_3}/x_{n_1}$ and $c_1 = 1$. The two outcomes are $|W\rangle$ with probability $3x_{n_2}x_{n_3}/x_{n_1}$ or $|\Psi^{(n_1 n_2)}\rangle$. On $|W\rangle$, the Fortescue-Lo Protocol of success probability of at least $\epsilon(3x_{n_2}x_{n_3}/x_{n_1})^{-1}$ is performed. This procedure yields an overall EPR probability of at least $\kappa(\vec{x}) - \epsilon$.

Now to achieve exactly probability $\kappa(\vec{x})$, a party with maximal component value can never make a measurement by Theorem 1. But suppose (without loss of generality) we begin with a state whose components satisfy $x_A \geq x_B \geq x_C$. Then by decomposing the transformation into a sequence of weak measurements, in order for parties B and C to end up maximally entangled, one branch of LOCC must pass through a state $\vec{y}$ for which y_A equals y_B or y_C, and a party with maximal component value will perform the next measurement. The only other option to avoid this scenario is that $p_{BC} = 0$ for the transformation. But then by Theorem 1, the maximum success probability is $\eta(\vec{x}) < \kappa(\vec{x})$. ■

As an immediate corollary of Theorem 2, we can now find convergent sequences of LOCC transformations whose limit cannot be performed by LOCC. Specifically, the Fortescue-Lo protocols [28] provide a sequence of LOCC maps $(\mathcal{L}_n)_{n=1,\ldots}$ that respectively achieve transformation (1) with probability $p_{AB} + p_{AC} + p_{BC} > 1 - \frac{1}{n}$ for initial state $|W\rangle$. Thus, the limit transformation $\mathcal{L}$ obtains $p_{AB} + p_{AC} + p_{BC} = 1$ which, by Theorem 2, is not LOCC achievable. Note this argument holds for both finite and infinite round protocols, with the latter known to be strictly more powerful [17].

We next turn to the question of how well SEP can perform transformation (1). For a three-qubit W-class state $\sqrt{x_A}|100\rangle + \sqrt{x_B}|010\rangle + \sqrt{x_C}|001\rangle$ with $1/2 \geq x_A \geq x_B \geq x_C$, the following separable measurement operators randomly distill the state with probability one:

$$\begin{aligned}
M_{AB} &= (\lambda_{CB}|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (\lambda_{CA}|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes |0\rangle\langle 0| \\
M_{AC} &= (\lambda_{BC}|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes |0\rangle\langle 0| \otimes (\lambda_{BA}|0\rangle\langle 0| + |1\rangle\langle 1|) \\
M_{BC} &= |0\rangle\langle 0| \otimes (\lambda_{AC}|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (\lambda_{AB}|0\rangle\langle 0| + |1\rangle\langle 1|) \\
M_{000} &= (1 - \lambda_{CA}^2\lambda_{CB}^2 - \lambda_{BA}^2\lambda_{BC}^2 - \lambda_{AB}^2\lambda_{AC}^2)^{1/2}|000\rangle\langle 000| \\
M_{111} &= |111\rangle\langle 111|,
\end{aligned} \tag{7}$$

where $\lambda_{ij} = \sqrt{\frac{1-2x_i}{2x_j}}$. As an example, consider the distillation rates on the one parameter family of states $|\psi_s\rangle = \sqrt{s}|100\rangle + \sqrt{\frac{1-s}{2}}(|010\rangle + |001\rangle)$ for $\frac{1}{3} \leq s \leq \frac{1}{2}$. The LOCC optimal probability is given by $\kappa(\psi_s) = 2(1-s) - (1-s)^2/(4s)$. A comparison of SEP and LOCC is depicted in Fig. 1.

In this Letter, we have studied the random distillation of three-qubit W-class states by separable operations and LOCC. We have shown that the limit of Fortescue-Lo Protocols cannot be implemented by LOCC, and therefore in general one must use caution when speaking of "optimal LOCC protocols." We have shown a 12.5% difference between LOCC and SEP in terms of success probability, which is orders of magnitude larger than any previous quantification of the operational gap. Extensions of these results to multi-partite systems will be presented in forthcoming manuscripts [35,36].

Experimentally, the protocol in Theorem 2 could possibly be demonstrated with today's technology due to the relative ease in generating W-type entanglement [37,38]. Furthermore, specific setups for implementing random distillation using ion traps have already been proposed [39].

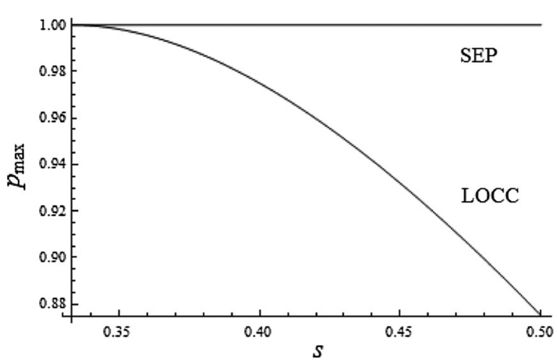

FIG. 1. LOCC vs SEP for the maximum probability of obtaining an EPR pair between any two parties as a function of s when the initial state is $\sqrt{s}|100\rangle + \sqrt{(1-s)/2}(|010\rangle + |001\rangle)$. The LOCC probability is $2(1-s) - (1-s)^2/4s$. A gap of 12.5% exists between SEP and LOCC.

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