Complete $O(\alpha_s^4)$ QCD Corrections to Hadronic Z Decays

P. A. Baikov
Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, 1(2), Leninskie gory, Moscow 119234, Russian Federation

K. G. Chetyrkin,* J. H. Kühn, and J. Rittinger
Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany
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Corrections of order α_s^4 for the axial singlet contributions for the decay rate of the Z boson into hadrons are evaluated in the limit of the heavy top quark mass. Combined with recently finished $\mathcal{O}(\alpha_s^4)$ calculations of the nonsinglet corrections, the new results directly lead us to the first *complete* $\mathcal{O}(\alpha_s^4)$ prediction for the total hadronic decay rate of the Z boson. The new $\mathcal{O}(\alpha_s^4)$ term in Z-decay rate leads to a significant stabilization of the perturbative series, to a reduction of the theory uncertainty in the strong coupling constant α_s, as extracted from these measurements, and to a small shift of the central value.

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The precise determination of the Z-boson decay rate into hadrons at LEP [1] has led to one of the most precise determinations of the strong coupling constant α_s. From the experimental side, in view of the fully inclusive nature of this measurement, the result is fairly robust, in particular since it is insensitive to simulations of the hadronic final state. Hence, the error is essentially dominated by the statistical uncertainty. From the theory side, the advantage of the measurement is its high energy, and as a result, the irrelevance of nonperturbative and power-law suppressed terms. The smallness of α_s at high energies then leads to a rapid decrease of higher order corrections in the perturbative series and, correspondingly, to a significant reduction of the theory error.

A variety of methods has been suggested to estimate the remaining uncertainty in the theory prediction. Using the last calculated term is probably the most conservative approach; varying the renormalization scale μ within an energy range characteristic for the problem (e.g., $M_Z/3 < \mu < 3M_Z$) is frequently used, albeit with considerable ambiguity in the actual choice of the region of the μ-variation. In order to reduce the theoretical uncertainty in the extraction of α_s to a level significantly smaller than the experimental one (which amounts to ± 0.0026 at present [1]), the knowledge of the corrections of $\mathcal{O}(\alpha_s^4)$ is necessary. At the same time this calculation opens the window for a considerable improvement in the α_s-determination at *Giga Z*, the project of a high-luminosity linear collider operating at the Z-resonance (see, e.g., [2], where a precision of 0.0005 to 0.0007 has been advertised). The dominant part of the α_s^4-corrections, the "nonsinglet" piece, has been evaluated in [3]. This has lead to a slight shift of the central value of α_s upward from 0.1185 ± 0.0026 to 0.1190 ± 0.0026 [3] and a reduction of the theory error far below the error of 0.0026 from the experiment. However, as noted already in [3], for a complete evaluation of the decay rate in $\mathcal{O}(\alpha_s^4)$, an additional set of corrections, namely those for the "singlet" contributions, is required. For the axial current correlator, these start at $\mathcal{O}(\alpha_s^2)$ [4,5], and for the vector correlator at $\mathcal{O}(\alpha_s^3)$. Both of them are presently known to third order in α_s only [6–10]. Hence, for a completely consistent $\mathcal{O}(\alpha_s^4)$ extraction of the strong coupling, the extension of these results by one order in α_s is required.

Before describing this calculation in detail, let us briefly recall the basic structure of QCD corrections to the correlator of the electromagnetic and the neutral current, respectively, their similarities and their main differences. After splitting off inessential kinematic factors, the absorptive part of the current-current correlator of the electromagnetic current is expressed by the familiar R-ratio

$$R^{\text{em}} = 3\Big[\sum_f q_f^2 r_{\text{NS}}^V + \Big(\sum_f q_f\Big)^2 r_S^V\Big], \tag{1}$$

where r_{NS}^V and r_S^V stand for the (numerically dominant) nonsinglet and the singlet part respectively. The corresponding decomposition for the correlator of the neutral current involves the following four terms

$$R^{\text{nc}} = 3\Big[\sum_f v_f^2 r_{\text{NS}}^V + \Big(\sum_f v_f\Big)^2 r_S^V + \sum_f a_f^2 r_{\text{NS}}^A + r_{S;t,b}^A\Big], \tag{2}$$

with $v_f \equiv 2I_f - 4q_f s_W^2$, $a_f \equiv 2I_f$ and s_W defined as effective weak mixing angle. Here, all but the top quark are assumed to be massless.

(Mass corrections to both vector and axial vector correlator due to other massive quarks are dominated by the bottom quark and can be classified by orders in m_b^2/M_Z^2 and α_s. Up to $\mathcal{O}(\alpha_s^2 m_b^2/M_Z^2)$ and $\mathcal{O}(\alpha_s^2 m_b^4/M_Z^4)$, they can be found in [11], as can terms of order $\alpha_s^2 m_b^2/M_Z^2$ (const + $\log m_b^2/M_Z^2$) and $\alpha_s^2 m_b^2/M_t^2$ (const + $\log m_b^2/M_Z^2$) that arise from axial vector singlet contributions. Terms of order

$\alpha_s^3 m_b^4/M_Z^4$ and $\alpha_s^4 m_b^2/M_Z^2$ can be found in [12,13], respectively. Corrections of order $\alpha_s^2 m_Z^2/m_t^2$ and $\alpha^3 m_Z^2/m_t^2$ from singlet and nonsinglet terms are known from [4,5,14,15], respectively. These are important for the actual α_s-determination but will not be discussed further in the present Letter.)

From the prefactors of the nonsinglet contributions in electromagnetic, vector, and axial correlator, it is evident that different quark flavors contribute incoherently, hence additive to the rate. Thus their contribution is significantly enhanced in comparison with the singlet terms where amplitudes from different flavors interfere destructively, with prefactors $(\sum_f q_f)^2$ and $(\sum_f v_f)^2$ for the electromagnetic and neutral current respectively.

Nonsinglet contributions are present at the parton level and the QCD corrections are known in second [16], third [6,7], and fourth [3] order in α_s. In terms of Feynman diagrams, nonsinglet contributions are characterized by the fact that one quark loop connects the two external currents [Fig. 1(a)]. In the absorptive part of this fermion loop, no top quark is present due to kinematic reasons, whence the nonsinglet functions are identical $r_{\rm NS}^V = r_{\rm NS}^A \equiv r_{\rm NS}$.

In the case of singlet contributions of the vector current, the two currents couple to two different quark loops [Fig. 1(b)] requiring a three-gluon intermediate state. Correspondingly, the leading term is of $\mathcal{O}(\alpha_s^3)$ and has been obtained long ago [6,7]. The next-to-leading-order (NLO) corrections to this result are of $\mathcal{O}(\alpha_s^4)$. They serve to soften the strong scale dependence of the $\mathcal{O}(\alpha_s^3)$ result and stabilize the theory prediction; they will be the subject of this Letter.

The situation is different in the case of the singlet axial vector current correlator. The axial couplings of the two members of an isospin doublet are opposite equal. Hence, their singlet contribution vanishes, if the corresponding quark masses are equal. This approximation is valid for the two lightest quark doublets. The only remaining contribution originates from the combination of bottom and top quarks with their specific mass hierarchy $m_b^2 \ll M_Z^2 \ll m_t^2$ [Fig. 1(c)]. In this case, the contribution starts at $\mathcal{O}(\alpha_s^2)$ and is further enhanced by the "large" logarithm $\log(m_t^2/M_Z^2)$ [4,5]. Corrections of $\mathcal{O}(\alpha_s^3)$ have been calculated in [8–10]; those of $\mathcal{O}(\alpha_s^4)$ will be the subject of this Letter.

The evaluation of the NLO terms of r_V^S requires the calculation of the absorptive parts of five-loop diagrams with massless propagators which, with the help of some

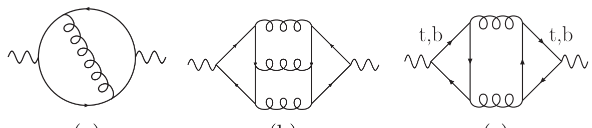

FIG. 1. Different contributions to r-ratios: (a) nonsinglet, (b) vector singlet, and (c) axial vector singlet.

complicated combinatorics based on the R^*-operation [17], can be boiled down to the calculation of four-loop propagator diagrams. The latter have been computed via reduction to 28 master integrals, based on evaluating sufficiently many terms of the $1/D$ expansion [18] of the corresponding coefficient functions [19]. This direct procedure required huge computing resources and was performed using a parallel version [20] of FORM [21]. The master integrals are reliably known from [22–24]. The details of the calculation, the results in analytic form, and their relation to the Gross–Llewellyn Smith sum rule will be given in [25].

The evaluation of the next-to-next-to-leading-order terms of $R_{S;t,b}^A$ involves again absorptive parts of five-loop diagrams with massless propagators, however, in addition also absorptive parts of four-loop diagrams combined with one-loop massive tadpoles, etc. down to one-loop massless diagrams together with four-loop massive tadpoles. The latter have been computed with the help of the Laporta algorithm [26] implemented in CRUSHER [27]. The methods employed in our calculations, together with the results, will be described in more detail in [25].

The result is valid in the limit $M_Z^2 \ll 4M_t^2$, an excellent approximation as evident from the lower orders. The relative importance of the various terms is best seen from the results for the various r-ratios introduced above, expressed in numerical form

$$\begin{aligned} r_{\rm NS} &= 1 + a_s + 1.4092a_s^2 - 12.7671a_s^3 - 79.9806a_s^4, \\ r_S^V &= -0.4132a_s^3 - 4.9841a_s^4, \\ r_{S:t,b}^A &= (-3.0833 + l_t)a_s^2 + (-15.9877 + 3.7222l_t \\ &\quad + 1.9167l_t^2)a_s^3 + (49.0309 - 17.6637l_t + 14.6597l_t^2 \\ &\quad + 3.6736l_t^3)a_s^4, \end{aligned} \tag{3}$$

with $a_s = \alpha_s(M_Z)/\pi$ and $l_t = \ln(M_Z^2/M_t^2)$. Since all three r-ratios are separately scale invariant, the corresponding results for a generic value of $\alpha_s(\mu)$ can easily be reconstructed. Using for the pole mass M_t the value 172 GeV, the axial singlet contribution is given in numerical form by

$$r_{S;t,b}^A = -4.3524a_s^2 - 17.6245a_s^3 + 87.5520a_s^4. \tag{4}$$

Collecting now all QCD terms, the decay rate of the Z boson into hadrons can be cast into the following form

$$\Gamma_Z = \Gamma_0 R^{\rm nc} = \frac{G_F M_Z^3}{24\pi\sqrt{2}} R^{\rm nc}. \tag{5}$$

Here, all electroweak corrections are assumed to be collected in the prefactor Γ_0, and the forementioned mass corrections are ignored as well as electroweak and mixed QCD-electroweak corrections [28–30]. Thus the R ratio is now known up to $\mathcal{O}(a_s^4)$:

$$R^{\rm nc} = 20.1945 + 20.1945 a_s + (28.4587 - 13.0575 + 0) a_s^2 + (-257.825 - 52.8736 - 2.12068) a_s^3 + (-1615.17 + 262.656 - 25.5814) a_s^4, \quad (6)$$

with $s_W^2 = 0.231$. The three terms in the parentheses display separately nonsinglet, axial singlet, and vector singlet contributions.

Let us now evaluate the impact of the newly calculated terms on the α_s-determination from Z-decays. Following our approach for the nonsinglet terms (where a shift $\delta\alpha_s = 0.0005$ had been obtained [3], consistent with an analysis [31] based on results of the Electroweak Working Group [1] and a modified interface to ZFITTER v. 6.42 [32,33] and confirmed by the G-fitter Collaboration [32–34]), we consider the quantity $R^{\rm nc}$ as "pseudo-observable." With a starting value $R^{\rm nc} = 20.9612$, if evaluated for $\alpha_s = 0.1190$ and without the α_s^4 singlet terms, a shift $\delta\alpha_s = -0.00008$ is obtained after including the newly calculated contributions.

As discussed in [3], the nonsinglet α_s^4 term leads to a considerable stabilization of the theory prediction and, correspondingly, to a reduction of the theory error. A similar statement holds true for the singlet contribution. To illustrate this aspect, the dependence on the renormalization scale μ is shown in Fig. 2 for $r_{\rm NS}$, r_S^V and $r_{S;t,b}^A$. The relative variation is significantly reduced in all three cases. In particular for the vector singlet case, we observe a shift of the result by about a factor 1.45 (for $\mu = M_Z$) and a considerable flattening of the result. Using for example the principle of minimal sensitivity [35] as a guidance for the proper choice of scale, $\mu = 0.3M_Z$ seems to be favored, leading to an amplification of the LO result by a factor 1.68 (if the latter is evaluated for $\mu = M_Z$, as done traditionally).

Let us assume that the remaining theory uncertainties from $r_{\rm NS}$, r_S^V and $r_{S;t,b}^A$ can be estimated by varying μ between $M_Z/3$ and $3M_Z$ and using the maximal variation as twice the uncertainty δr. This leads to $\delta\Gamma_{\rm NS} = 0.101$ MeV, $\delta\Gamma_S^V = 0.0027$ MeV and $\delta\Gamma_S^A = 0.042$ MeV. Even adding these terms linearly, they are far below the experimental error of $\delta\Gamma_{\rm exp} = 2.0$ MeV [36]. In combination with the quadratic and quartic mass terms, which are known to $\mathcal{O}(\alpha_s^4)$ and $\mathcal{O}(\alpha_s^3)$ respectively, this analysis completes the QCD corrections to the Z decay rate.

Let us also comment on the impact of the α_s^4 singlet result on the measurement of $R^{\rm em}$ at low energies, i.e., in the region accessible at Beijing Spectrometer-III or at B factories, say between 3 and 10 GeV. Considering the large luminosities collected at these machines, a precise α_s determination from $R^{\rm em}$ seems possible [37]. In the low energy region, only r_S^V and $r_{\rm NS}^V$ contribute. Since $\sum_{f=u,d,s} q_f = 0$, the singlet contribution vanishes in the three-flavor case. If we consider the region above charm and below bottom threshold, say at 10 GeV, only u, d, s,

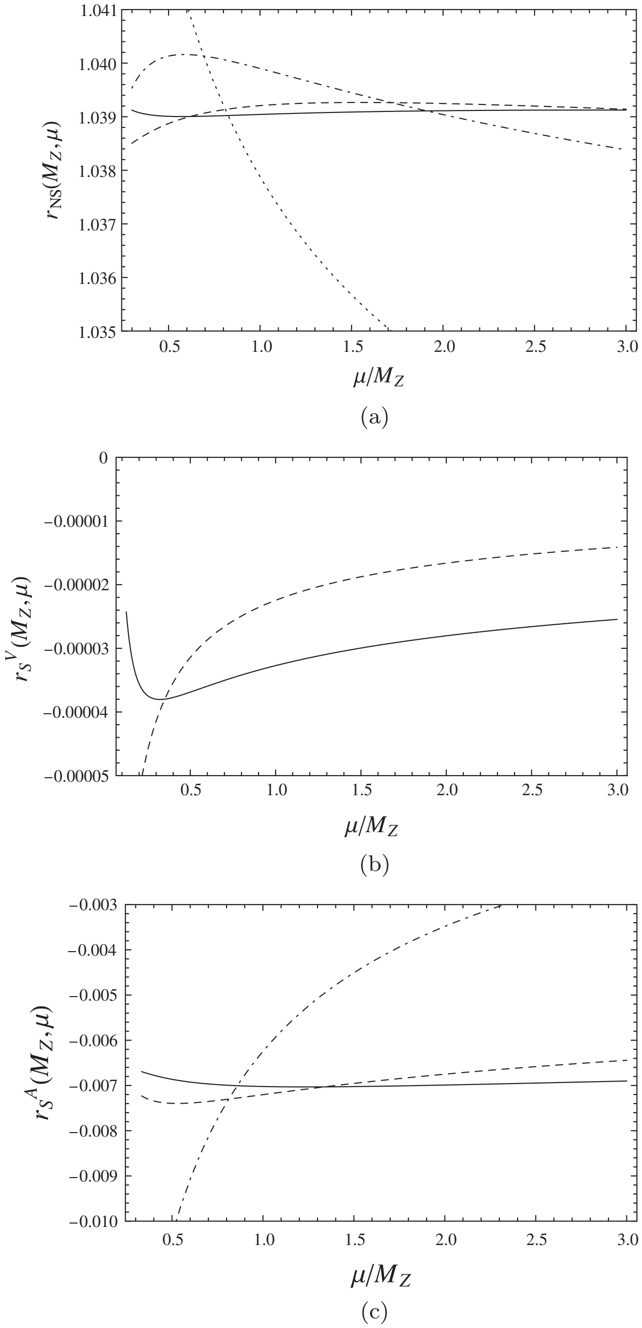

FIG. 2. Scale dependence of (a) non-singlet $r_{\rm NS}$, (b) vector singlet r_S^V, and (c) axial vector singlet $r_{S;t,b}^A$. Dotted, dash-dotted, dashed, and solid curves refer to $\mathcal{O}(\alpha_s)$ up to $\mathcal{O}(\alpha_s^4)$ predictions. $\alpha_s(M_Z) = 0.1190$ and $n_l = 5$ is adopted in all these curves.

and c quarks contribute, the relative weight of the r_S^V in Eq. (1) is given by $(\sum q_f)^2/(\sum q_f^2) = 2/5$, and thus is fairly suppressed. At energy of 10 GeV, in the absence of open bottom quark contribution, it seems appropriate to analyze the results in an effective four-flavor theory with

$$r_S^V = -0.41318 a_s^3(\mu) - (5.1757 + 2.5824 \ln\mu^2/s) a_s^4(\mu).$$

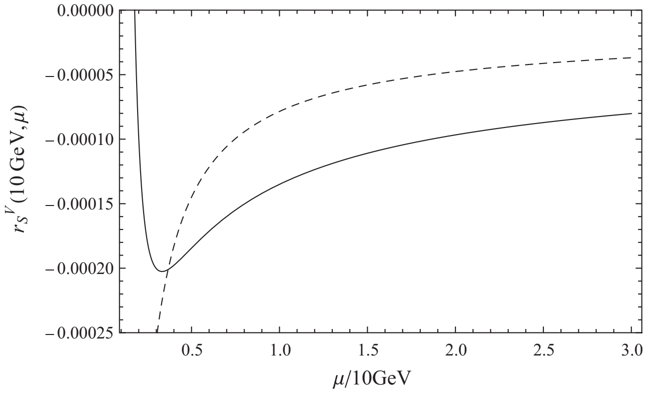

FIG. 3. Scale dependence of the vector singlet r_S^V around 10 GeV. Dashed and solid curves refer to $\mathcal{O}(\alpha_s^3)$ and $\mathcal{O}(\alpha_s^4)$ predictions. $n_l = 4$ and $\alpha_s(10\text{ GeV}) = 0.1806$ as obtained with the use of package RUNDEC [41] have been assumed.

As shown in Fig. 3, it is evident that the scale dependence is softened in NLO. Again a scale μ around $0.3\sqrt{s}$ minimizes the NLO corrections.

In conclusion, we want to mention that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using parallel MPI-based [20] as well as thread-based [38] versions of FORM [21]. For evaluation of color factors we have used the FORM program COLOR [39]. The diagrams have been generated with QGRAF [40].

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*Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.

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