Nonlinear Modes in Finite-Dimensional $\mathcal{PT}$-Symmetric Systems

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By rearrangements of waveguide arrays with gain and losses one can simulate transformations among parity-time ($\mathcal{PT}$-) symmetric systems not affecting their pure real linear spectra. Subject to such transformations, however, the nonlinear properties of the systems undergo significant changes. On an example of an array of four waveguides described by the discrete nonlinear Schrödinger equation with dissipation and gain, we show that the equivalence of the underlying linear spectra does not imply similarity of the structure or stability of the nonlinear modes in the arrays. Even the existence of one-parametric families of nonlinear modes is not guaranteed by the $\mathcal{PT}$ symmetry of a newly obtained system. In addition, the stability is not directly related to the $\mathcal{PT}$ symmetry: stable nonlinear modes exist even when the spectrum of the linear array is not purely real. We use a graph representation of $\mathcal{PT}$-symmetric networks allowing for a simple illustration of linearly equivalent networks and indicating their possible experimental design.

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The effect of dissipation or gain on dynamics of a physical system is a fundamental issue in both classical and quantum theories. Optics is one of the areas where the respective models appear naturally and have been explored for many years in the context of different kinds of dissipative solitons [1]. One of a number of widely used, fundamental, and simple models is an array of waveguides in the presence of gain and losses [2]. This model is described by the discrete nonlinear Schrödinger equation (DNLSE), which is fairly general. Its applications range from the so-called discrete optics [3] to biophysics [4] and the mean field theory of Bose-Einstein condensates [5] (for a broad range of applications of DNLSE, see also [6]).

Recently, great interest in systems with dissipation and gain was triggered by the discovery of the so-called parity-time ($\mathcal{PT}$) potentials, which in a definite range of parameters obey purely real spectrum [7]. Numerous linear physical systems for which $\mathcal{PT}$ symmetry is of great relevance have been proposed. Among them we mention non-Hermitian extension of quantum mechanics [8], electromagnetic wave propagation in a planar waveguide filled with active media [9], and beam propagation in optical lattices [10]. The phenomenon of $\mathcal{PT}$ symmetry breaking has been experimentally implemented in optics [11], where the equations governing the system were earlier known as describing a unidirectional coupler, i.e., as a particular form of the DNLSE [2].

Nonlinear $\mathcal{PT}$-symmetric problems were first posed in the context of the quantum field theory accounting for cubic interactions [12] and in guided wave theory [13]. Being natural for optical applications, the nonlinear problems received particular attention in the context of existence of gap solitons [13] and defect modes [14] in $\mathcal{PT}$-symmetric lattices, as well as in context of the nonlinear $\mathcal{PT}$-symmetric couplers in stationary [15,16] and solitonic [17] regimes. More generally, the nonlinearity enriches possible statements of the problem allowing for including the effects on nonlinear [18], as well as both linear and nonlinear [19] $\mathcal{PT}$-symmetric potentials.

It is known that by applying a similarity transformation to a given linear $\mathcal{PT}$-symmetric system, a new system with real spectrum can be constructed. Thus in [20] new potentials (not necessarily $\mathcal{PT}$ symmetric) with real spectra were constructed using the Darboux transformation, while in [21] pseudo-Hermitian operators were introduced and unitary equivalence of $\mathcal{PT}$-symmetric and Hermitian operators was established. It turns out, however, that possible mutual reductions of Hermitian, $\mathcal{PT}$-symmetric, and pseudo-Hermitian linear operators leaving the spectrum pure real, may introduce dramatic changes in the properties of the respective nonlinear systems. The analysis of such changes is the main goal of the present Letter.

More specifically, we show that $\mathcal{PT}$-symmetric systems obeying the same linear spectrum may either have one-parametric families of nonlinear modes or have not. If the families exist, stability of the modes is essentially different for different systems, still having the same linear spectrum. Moreover, stable nonlinear modes may exist beyond the $\mathcal{PT}$ symmetry breaking. For a discrete system consisting of four waveguides we find that breaking of $\mathcal{PT}$ symmetry can occur in two different ways: the linear spectrum acquires either two complex and two real eigenvalues, or all four eigenvalues become complex. Finally, we represent each underlying linear system by a graph, allowing one to catalog different linearly equivalent $\mathcal{PT}$-symmetric systems.

We consider an array of N waveguides (sites) and denote the field in the nth waveguide by $q_n(z)$, where z is the propagation distance. If each waveguide have dissipation or gain described by γ_n, positive or negative, respectively, then the field propagation is governed by the DNLSE

$$i\dot{q}_n = -\sum_{m=1}^{N} K_{nm}q_m - |q_n|^2 q_n - i\gamma_n q_n, \quad (1)$$

where $\dot{q}_n = dq(z)/dz$. Here we admit the existence of nonlocal coupling among the waveguides, which is described by the coefficients $K_{nm} = K_{mn} = K_{nm}^*$ which will be treated as entries of the real symmetric matrix $\mathbf{K}$: $\mathbf{K} = \mathbf{K}^\dagger$, where $\mathbf{K}^\dagger$ is the Hermitian conjugate matrix. It is convenient to introduce diagonal matrices $\mathbf{G} = \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$ and $\mathbf{F}(\mathbf{q}) = \mathrm{diag}(|q_1|^2, |q_2|^2, \ldots, |q_N|^2)$, which describe the dissipation and the nonlinear part of the system, respectively. Then the system (1) can be rewritten in the form

$$i\dot{\mathbf{q}} = -[\mathbf{H} + \mathbf{F}(\mathbf{q})]\mathbf{q}, \qquad \mathbf{H} = \mathbf{K} + i\mathbf{G}. \quad (2)$$

We search stationary nonlinear modes in the form $\mathbf{q}(z) = e^{ibz}\mathbf{w}$, where b is the propagation constant, and $\mathbf{w} = (w_1, w_2, \ldots, w_N)^T$ solves the stationary DNLSE

$$b\mathbf{w} = [\mathbf{H} + \mathbf{F}(\mathbf{w})]\mathbf{w}. \quad (3)$$

Requiring the spectrum of the linear problem $b\mathbf{w} = \mathbf{H}\mathbf{w}$ to be real, which is necessary for all linear modes to be propagating, we impose the constraint $\sum_{n=1}^{N}\gamma_n = 0$.

The matrix $\mathbf{H}$ is $\mathcal{PT}$ symmetric if it commutes with a $\mathcal{PT}$ operator: $[\mathcal{PT}, \mathbf{H}] = 0$. Hereafter $\mathcal{P}$ is an orthogonal symmetric (and therefore Hermitian) matrix, and $\mathcal{T}$ is elementwise complex conjugation: $\mathcal{T}\mathbf{q} = \mathbf{q}^*$. Using that $\mathbf{H}^\dagger = \mathcal{T}\mathbf{H}\mathcal{T} = \mathcal{P}\mathbf{H}\mathcal{P}$, we observe that the linear system $i\dot{\mathbf{q}} = -\mathbf{H}\mathbf{q}$ admits an integral of motion (see also [22]) $Q = \frac{1}{N}\langle\mathcal{P}\mathbf{q}, \mathbf{q}\rangle$, where the inner product is defined as $\langle\mathbf{u}, \mathbf{v}\rangle = \sum_{n=1}^{N} v_n^* u_n$.

Now we can specify the problem at hand: we consider the existence and stability of nonlinear modes of $\mathcal{PT}$-symmetric lattices whose linear parts are related to each other by similarity transformations, all having the nonlinearity of the on-site type. Such a statement is natural for arrays of optical waveguides, since linear links among them can be arranged by assembling waveguides in different geometries, while the dissipation or gain and the nonlinearity are the characteristics of each particular waveguide, which can be routinely controlled (see also Fig. 3). One of our main findings is that linearly equivalent $\mathcal{PT}$-symmetric lattices result in qualitatively different properties of their nonlinear extensions.

$\mathcal{PT}$-symmetric "quadrimer".—Since the gain and dissipation must compensate each other, the simplest models allowing for nontrivial distribution of the dissipation have three or four waveguides. Below we concentrate on a quadrimer, respectively, setting $N = 4$. We start by revisiting recently considered in [16] system with the next-neighbor interactions: $K_{nm} = \delta_{|n-m|,1}$. The corresponding matrix, which we denote as $\mathbf{H}_0$, is $\mathcal{PT}$ symmetric with respect to

$$\mathcal{P}_0 = \begin{pmatrix} \mathbf{0} & \sigma_1 \\ \sigma_1 & \mathbf{0} \end{pmatrix}$$

(hereafter $\sigma_{1,2,3}$ are the Pauli matrices and $\mathbf{0}$ is the 2×2 zero matrix). Depending on particular values of $\gamma_{1,2}$, three different situations are possible: (i) unbroken or *exact* $\mathcal{PT}$ symmetry, when all the eigenvalues $\tilde{b}_n$, $n = 1, \ldots, 4$, of $\mathbf{H}_0$ are real; (ii) broken $\mathcal{PT}$ symmetry with two real and two complex conjugated eigenvalues (notice that this is possible only if $\gamma_1 \neq \gamma_2$); (iii) broken $\mathcal{PT}$ symmetry with all $\tilde{b}_n$ complex. Thus, the "phase space" (γ_1, γ_2) can be divided into three domains as it is shown in the phase diagram (PD) of Fig. 1. A feature of the phase diagram is the existence of the triple points T_j, $j = 1, \ldots, 4$, where the three domains touch. The triple points correspond to values $\gamma_{1,2}$ for which $\tilde{b}_n = 0$ for each $n = 1, \ldots, 4$. Depending on how $\gamma_{1,2}$ change in vicinity of T_j, either the $\mathcal{PT}$-symmetric phase or one of the $\mathcal{PT}$ symmetry broken phases arise.

If $\mathbf{H}_0$ is exactly $\mathcal{PT}$ symmetric, then its linear eigenstates $\tilde{\mathbf{w}}$ are simultaneously the eigenstates of the corresponding $\mathcal{PT}$ operator, i.e., $\mathcal{P}_0\mathcal{T}\tilde{\mathbf{w}} = \tilde{\mathbf{w}}$ (up to irrelevant phase shift). It is natural to look for nonlinear modes that possess the same property: $\mathcal{P}_0\mathcal{T}\mathbf{w} = \mathbf{w}$. Therefore we require $w_1 = w_4^*$, $w_2 = w_3^*$, which reduces Eq. (3) to

$$bw_1 = w_2 + |w_1|^2 w_1 + i\gamma_1 w_1, \quad (4a)$$

$$bw_2 = (w_1 + w_2^*) + |w_2|^2 w_2 + i\gamma_2 w_2. \quad (4b)$$

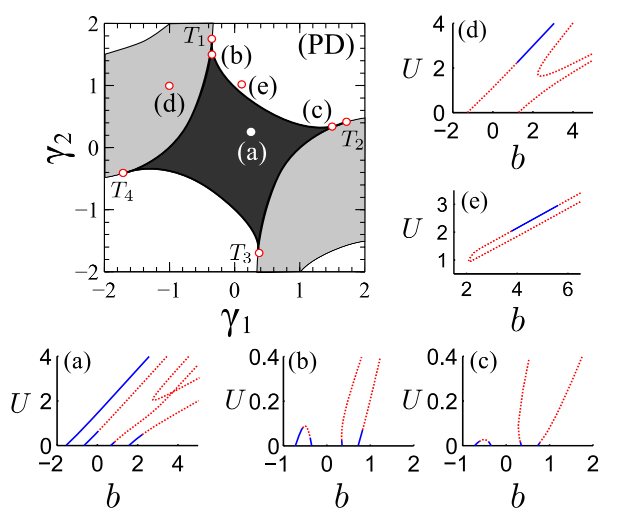

FIG. 1 (color online). "Phase diagram" (PD) for the linear quadrimer $\mathbf{H}_0$. The dark-gray diamond-shaped domain corresponds to unbroken $\mathcal{PT}$ symmetry; in the light-gray domains there are two real and two complex eigenvalues. In the white domains all eigenvalues are complex. In panels (a)–(e), corresponding to the points (a)–(e) in the panel (PD), we show families of nonlinear modes for: (a) $\gamma_{1,2} = 0.25$; (b) $\gamma_1 \approx -0.37$, $\gamma_2 \approx 1.49$; (c) $\gamma_1 \approx 1.49$, $\gamma_2 \approx 0.36$; (d) $\gamma_{1,2} = \mp 1$; (e) $\gamma_1 = 0.1$, $\gamma_2 = 0.95$. Stable (unstable) modes are shown by solid blue (dashed red) lines.

We represent $w_1 = W_1 e^{i\Phi}$, where W_1 and Φ are real. Then Eq. (4a) gives $w_2 = W_2 e^{i\Phi}$, where $W_2 = W_1(b - W_1^2 - i\gamma_1)$ is complex, and from Eq. (4b) we obtain $e^{-2i\phi} = f(W_1)$, where $f(W_1) = (bW_2 - W_1 - |W_2|^2 W_2 - i\gamma_2 W_2)/W_2^*$. If a root of the equation $|f(W_1)|^2 = 1$ is found, then w_1 and w_2 can be readily obtained. Thus nonlinear modes of the quadrimer correspond to the roots of a single equation $|f(W_1)|^2 = 1$ with respect to one real unknown W_1. It is a purely technical matter to reduce the latter equation to: $P_8(W_1^2) = 0$, where $P_8(\xi)$ is an eighth-degree polynomial with real coefficients. Each positive root of $P_8(\xi)$ corresponds to a nonlinear mode of the quadrimer. Since the roots depend continuously on b, the nonlinear modes constitute continuous families for fixed parameters of the system [23]. As it is customary, such families can be represented on the plane (U, b) where $U = \frac{1}{4}\sum_{n=1}^{4} |w_n|^2$ is the total energy flow in the array. Panels (a)–(e) of Fig. 1 illustrate typical examples of the families, as well as linear stability of the modes. When $\gamma_{1,2}$ belong to the domain of unbroken $\mathcal{PT}$ symmetry [see Fig. 1(a)], one observes four families branching off from the linear limit, i.e., from the points $b = \tilde{b}_n$, $U = 0$. In Fig. 1(a) there also exist families that can not be continued from the linear limit. In panels (b) and (c) we also address the points that belong to the domain of unbroken $\mathcal{PT}$ symmetry but are situated closely to the triple points $T_{1,2}$. In these panels one observes that after the bifurcation from the linear limit, all four families rapidly lose stability and two of them cease to exist if U is sufficiently large. Comparing panel (a) with panels (b) and (c), we conclude that increase of $\gamma_{1,2}$, i.e., approaching the $\mathcal{PT}$ symmetry breaking boundary, is unfavorable for existence and stability of the modes. However, the most surprising fact is that stable nonlinear modes can be found in the domains of broken $\mathcal{PT}$ symmetry. Both in panel (d), which addresses the case when the spectrum consists of two real and two complex eigenvalues, and in panel (e), i.e., when all the eigenvalues are complex, one can find stable modes.

Hermitian quadrimer.—If $\mathbf{H}_0$ is exactly $\mathcal{PT}$ symmetric, then there exists a unitary matrix $\mathbf{R}$, which transforms $\mathbf{H}_0$ to a Hermitian matrix $\mathbf{H}_H$ [21]: $\mathbf{R}\mathbf{H}_0\mathbf{R}^{-1} = \mathbf{H}_H = \mathbf{H}_H^\dagger$. This means that in the linear limit the modes in the array with gain and losses described by $\mathbf{H}_0$ have the same propagation constants as the modes in the array without gain and losses, which is described by $\mathbf{H}_H$. Hence, for any $\gamma_{1,2}$ lying in the domain of unbroken $\mathcal{PT}$ symmetry of $\mathbf{H}_0$, one can introduce a new DNLSE $i\dot{\mathbf{q}} = -[\mathbf{H}_H + \mathbf{F}(\mathbf{q})]\mathbf{q}$ [cf. (2)]. Following [21], one can find $\mathbf{H}_H$ explicitly and observe that all its nonzero elements are real and given by $\mathbf{H}_{H,12} = \mathbf{H}_{H,21} = h_1$, $\mathbf{H}_{H,14} = \mathbf{H}_{H,41} = h_2$, $\mathbf{H}_{H,23} = \mathbf{H}_{H,32} = h_3$, with h_j being dependent on $\gamma_{1,2}$. By construction, the matrix $\mathbf{H}_H$ has the same eigenvalues as $\mathbf{H}_0$ for the given $\gamma_{1,2}$.

Unlike in the $\mathcal{PT}$-symmetric case, the modes of the nonlinear system with linear part described by $\mathbf{H}_H$ can be searched as real valued and either even or odd, i.e., solving the system $bw_1 = h_1 w_2 \pm h_2 w_1 + w_1^3$, $bw_2 = h_1 w_1 \pm h_3 w_2 + w_2^3$, where "+" ("−") stays for even (odd) modes. This system is equivalent to a fourth-degree polynomial equation with respect to w_1^2. Families of even and odd nonlinear modes of the Hermitian quadrimer are illustrated in Fig. 2. Comparing Figs. 1 and 2, we observe that even if the matrices $\mathbf{H}_0$ and $\mathbf{H}_H$ have the same eigenvalues, the respective nonlinear systems show considerable differences in the properties of modes. The most visible differences are (i) for the Hermitian system, the families bifurcating from the linear limit never close forming a saddle-node bifurcation [cf. panels (b) and (c) in Figs. 1 and 2], (ii) the leftmost family of the Hermitian system is always stable, and (iii) in general, stable nonlinear modes of Eq. (2) with $\mathbf{H}_0$ and $\mathbf{H}_H$ correspond to different values of the propagation constant b.

"Generalized" quadrimer.—Being of the dissipative nature, the considered above $\mathcal{PT}$-symmetric quadrimer with linear part described by $\mathbf{H}_0$ possesses a property, usually typical for conservative systems—for the given parameters of the system (intersite interactions $\mathbf{K}$ and dissipation $\gamma_{1,2}$) its nonlinear modes constitute continuous families rather than appear as isolated attractors. This peculiarity of nonlinear $\mathcal{PT}$-symmetric systems was reported in several studies [13,16,18,19]. Here we argue that existence of the continuous families of nonlinear modes is not a typical property of $\mathcal{PT}$-symmetric systems. Specifically, the nonlinear $\mathcal{PT}$-symmetric systems that admit the families of the modes appear as "isolated points" in a continuous set of generic $\mathcal{PT}$-symmetric systems.

To this end we focus on the particular case $\gamma_1 = \gamma_2 = \gamma$, i.e., $\mathbf{G} = \mathrm{diag}(\gamma, \gamma, -\gamma, -\gamma)$, and introduce an one-parametric family of matrices $\mathbf{H}_0(\beta) = \mathbf{K}_0(\beta) + i\mathbf{G}$ with

$$\mathbf{K}_0(\beta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \cos\beta & -\sin\beta \\ 0 & \cos\beta & \sin 2\beta & \cos 2\beta \\ 0 & -\sin\beta & \cos 2\beta & -\sin 2\beta \end{pmatrix},$$

and real parameter β. One can ensure that $\mathbf{H}_0(\beta)$ is $\mathcal{PT}$ symmetric with respect to

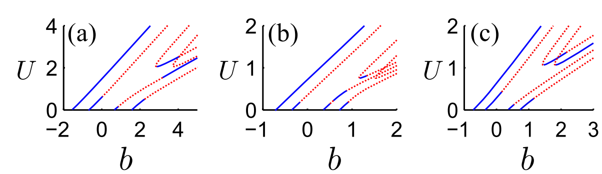

FIG. 2 (color online). Families of nonlinear modes of the Hermitian quadrimers, whose linear parts are described by $\mathbf{H}_H$, chosen to have the same linear eigenvalues $\tilde{b}_n$ as the corresponding $\mathcal{PT}$-symmetric quadrimers $\mathbf{H}_0$ illustrated in panels (a), (b), and (c) of Fig. 1.

$$\mathcal{P}_0(\beta) = \begin{pmatrix} \mathbf{0} & \rho(\beta) \\ \rho(\beta) & \mathbf{0} \end{pmatrix},$$

where $\rho(\beta) = \cos\beta\sigma_1 + \sin\beta\sigma_3$. For $\beta = 0$ the matrix $\mathbf{K}_0(0)$ includes only the next-neighbor interactions; i.e., $\mathbf{H}_0(0)$ is merely the linear part of the $\mathcal{PT}$-symmetric quadrimer studied above (with $\gamma_1 = \gamma_2$). The definition of $\mathbf{H}_0(\beta)$ guarantees that its eigenvalues do not depend on β. But the eigenvectors of $\mathbf{H}_0(\beta)$ do depend on β.

Next, using

$$\mathbf{M} = \begin{pmatrix} \mu & \mathbf{0} \\ \mathbf{0} & \mu \end{pmatrix},$$

where $\mu = \sigma_3 + i\sigma_2$, one can generate a new matrix $\mathbf{H}_1(\beta) = \mathbf{M}\mathbf{H}_0(\beta)\mathbf{M}^{-1}$, where $\mathbf{H}_1(\beta) = \mathbf{K}_1(\beta) + i\mathbf{G}$,

$$\mathbf{K}_1(\beta) = \begin{pmatrix} 1 & 0 & -k_- & k_+ \\ 0 & -1 & k_- & -k_+ \\ -k_- & k_- & \cos 2\beta & \sin 2\beta \\ k_+ & -k_+ & \sin 2\beta & -\cos 2\beta \end{pmatrix},$$

and $k_\pm = \frac{\sqrt{2}}{2}\sin(\beta \pm \frac{\pi}{4})$. Then $\mathbf{H}_1(\beta)$ is $\mathcal{PT}$ symmetric with respect to $\mathcal{P}_1(\beta) = \mathbf{M}\mathcal{P}_0(\beta)\mathbf{M}^{-1}$. Notice that the transformation $\mathbf{M}$ does not affect the dissipative component $i\mathbf{G}$, which is the same both for $\mathbf{H}_0(\beta)$ and $\mathbf{H}_1(\beta)$. Obviously, the eigenvalues of the matrix $\mathbf{H}_1(\beta)$ are the same as for $\mathbf{H}_0(\beta)$ and also do not depend on β.

To give better physical insight into the systems $\mathbf{H}_{0,1}(\beta)$, in Fig. 3 we introduce their weighted graph representation. The vertexes of graphs correspond to the sites q_n while the edges (lines) represent intersite coupling having weights equal to the values of the respective matrix elements: e.g., a line between the vertexes q_1 and q_2 corresponds to the elements $K_{1,2} = K_{2,1}$ of the matrix $\mathbf{K}$. Each vertex is supplied by the sign "+" or "−" corresponding to gain and dissipation. We notice that the loop edges, which describe the on-site interactions $K_{n,n}$, are not shown as being not relevant for the present consideration. The graph representation can be viewed also as indication on how one could place and connect the waveguides in an experiment in order to obtain the desirable $\mathcal{PT}$-symmeric quadrimer. It is worth noting that, say the bottom graph in left column [i.e., $\mathbf{H}_0(\pi/2)$] can be reshaped into the line distribution of the waveguides similar to the graph $\mathbf{H}_0(0)$.

Existence of nonlinear modes.—Turning to nonlinear properties of the arrays, whose linear links are described by $\mathbf{H}_{0,1}(\beta)$, let us suppose that the nth eigenstate of the underlying linear problem $b\mathbf{w} = \mathbf{H}_{0,1}(\beta)\mathbf{w}$ gives rise to a family of nonlinear modes. Then in the vicinity of the bifurcation point the nonlinear modes can be described using the expansion $\mathbf{w} = \varepsilon\tilde{\mathbf{w}}_n + o(\varepsilon)$, and $b = \tilde{b}_n + \varepsilon^2 b_n^{(2)} + o(\varepsilon^2)$, where ε is a small parameter, $\tilde{b}_n$ and $\tilde{\mathbf{w}}_n$ are the eigenvalue and the corresponding eigenvector of $\mathbf{H}_0(\beta)$ [or $\mathbf{H}_1(\beta)$]. The coefficient $b_n^{(2)}$ can be readily found: $b_n^{(2)} = \langle \mathbf{F}(\tilde{\mathbf{w}}_n)\tilde{\mathbf{w}}_n, \tilde{\mathbf{w}}_n^* \rangle / \langle \tilde{\mathbf{w}}_n, \tilde{\mathbf{w}}_n^* \rangle$. This means that the bifurcation of nonlinear modes is possible only if

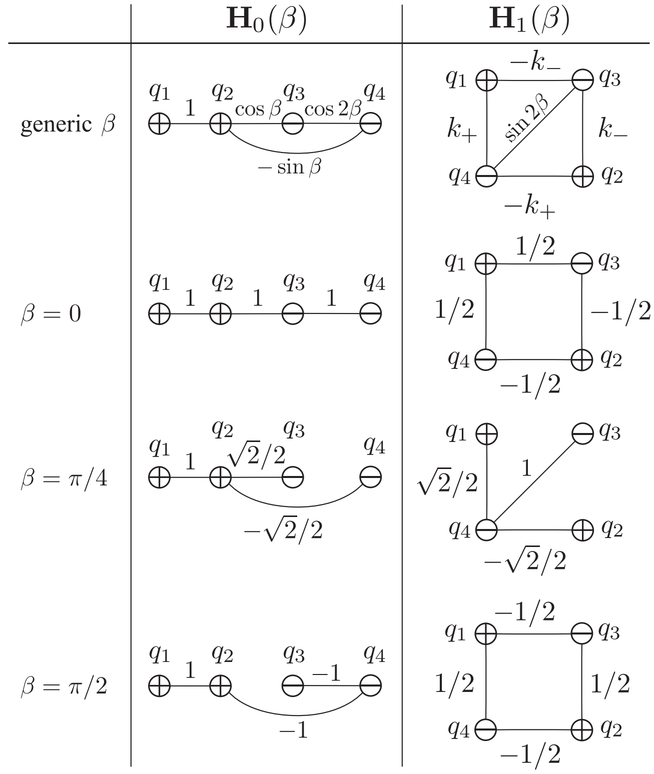

FIG. 3. Graph representation of the systems $\mathbf{H}_{0,1}(\beta)$ for generic and particular values of β. In terms of the optical applications, the circles with "+" [or "−"] represent waveguides with gain [or losses], while the lines indicate the directions, along which the coupling of the field has to be arranged.

$\mathrm{Im}b_n^{(2)} = 0$ for all n (we may conjecture that this condition is also sufficient for existence of the modes, what was observed in all our numerical simulations). The coefficient $b_n^{(2)}$ is easily computable. In Fig. 4 (left panel) $\mathrm{Im}b_n^{(2)}$ is plotted for $\mathbf{H}_0(\beta)$. Only at $\beta = \beta_k = \pi k/2$ the coefficient $b_n^{(2)}$ becomes real for all n and the system $\mathbf{H}_0(\beta)$ admits continuous families of nonlinear modes, while for all other β nonlinear modes bifurcating from the linear limit were not found.

To understand peculiarity of the values β_k we notice that relation $\mathcal{PT}\tilde{\mathbf{w}}_n = \tilde{\mathbf{w}}_n$ ensures that the denominator in the formula for $b_n^{(2)}$ is real for any β: $\langle \tilde{\mathbf{w}}_n, \tilde{\mathbf{w}}_n^* \rangle = \langle \mathcal{PT}\tilde{\mathbf{w}}_n, \mathcal{T}\tilde{\mathbf{w}}_n \rangle = \langle \mathcal{T}\tilde{\mathbf{w}}_n, \mathcal{PT}\tilde{\mathbf{w}}_n \rangle = \langle \tilde{\mathbf{w}}_n^*, \tilde{\mathbf{w}}_n \rangle$. Meanwhile $\langle \mathbf{F}(\tilde{\mathbf{w}}_n)\tilde{\mathbf{w}}_n, \tilde{\mathbf{w}}_n^* \rangle$ can have nonzero imaginary part. Then the

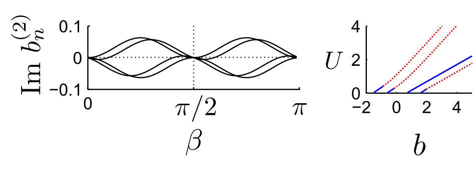

FIG. 4 (color online). $\mathrm{Im}b_n^{(2)}$ ($n = 1, \ldots, 4$) vs β for $\mathbf{H}_0(\beta)$ (left panel) and the families of modes of $\mathbf{H}_1(0)$ for $\gamma = 0.25$ (right panel).

reality of the coefficient $b_n^{(2)}$ is ensured by an additional constraint $\mathcal{PT}(\mathbf{F}(\tilde{\mathbf{w}}_n)\tilde{\mathbf{w}}_n) = \mathbf{F}(\tilde{\mathbf{w}}_n)\tilde{\mathbf{w}}_n$, which is satisfied only for $\beta = \beta_k$.

For the system $\mathbf{H}_1(\beta)$ the situation is similar—the families of nonlinear modes exist only for $\beta = \pi k/2$ where $\mathrm{Im} b_n^{(2)} = 0$ for all n. In the right panel of Fig. 4 we show families of nonlinear modes of the array whose linear part is described by $\mathbf{H}_1(0)$ with $\gamma = 0.25$. Comparing the latter panel with Fig. 1(a) (which also corresponds to $\gamma_{1,2} = 0.25$), we again notice that, whereas the corresponding arrays have the same eigenvalues in the linear limit, nonlinear modes of those arrays have essentially different properties.

To conclude, we have considered nonlinear properties of different $\mathcal{PT}$-symmetric lattices (discrete nonlinear Schrödinger equations with gain and dissipation), whose linear parts are related by similarity transformations preserving the spectrum. Such systems describe, in particular, arrays of optical waveguides with either gain or losses, which are properly arranged in the space. Alternatively, a physical realization of the described phenomenon is possible in arrays of Bose-Einstein condensates loaded in multiwell potentials, provided the atoms are eliminated from given wells and are condensed in the other wells, simulating in this way losses and gain.

On the case example of a $\mathcal{PT}$-symmetric quadrimer we have shown that the spectral equivalence of the underlying linear systems does not imply similarity of the nonlinear modes or their stability properties. We have found that the existence of one-parametric families of nonlinear modes is not guaranteed by the $\mathcal{PT}$ symmetry, and appears as a peculiarity of a system rather than a general property. It was also found that the stability of nonlinear modes is not directly related to the $\mathcal{PT}$ symmetry: stable nonlinear modes exist beyond the $\mathcal{PT}$ symmetry breaking threshold. If the system includes two different dissipative coefficients, then the "phase diagram" of the $\mathcal{PT}$-symmetric quadrimer allows for existence of "triple" points, where three different phases meet. Finally, we have shown that use of graph representation of $\mathcal{PT}$-symmetric networks gives straightforward indication on their possible experimental design in optics, and provides graphical illustration of linearly equivalent networks.

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[23] The continuous families of solutions presented herein are additional to the solutions found in [16] for $\gamma_1 = \gamma_2$, which required that the parameters of the system are interrelated. These two types of solutions are complementary in the complete set of possible standing wave solutions in this special case of equal values of the quadrimer gain or loss parameters.