Helical Superconducting Black Holes

Aristomenis Donos and Jerome P. Gauntlett

Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, United Kingdom
(Received 8 March 2012; published 22 May 2012)

We construct novel static, asymptotically five-dimensional anti–de Sitter black hole solutions with Bianchi type-VII$_0$ symmetry that are holographically dual to superconducting phases in four spacetime dimensions with a helical p-wave order. We calculate the precise temperature dependence of the pitch of the helical order. At zero temperature the black holes have a vanishing entropy and approach domain wall solutions that reveal homogenous, nonisotropic dual ground states with an emergent scaling symmetry.

DOI: 10.1103/PhysRevLett.108.211601 PACS numbers: 11.25.Tq, 04.50.Gh

Introduction.—The AdS/CFT correspondence is a powerful tool to analyze strongly coupled quantum field theories and there has been a surge of activity aimed at finding possible applications both to condensed matter systems and to QCD. One focus has been to holographically realize various kinds of phases via the construction of fascinating new classes of black hole solutions, which are also of interest in their own right.

An important development was the discovery of black brane solutions that are holographically dual to superconducting phases, or more precisely, superfluid phases [1]. These black holes carry a halo of charged hair that spontaneously breaks a global Abelian symmetry of the dual field theory. In the simplest examples, the charged hair is in the guise of a bulk scalar field, corresponding to a scalar order parameter in the dual field theory, and hence an s-wave superconducting phase. p-wave superconducting phases, in which the order parameter has angular momentum $l = 1$, have also been realized [2,3].

Spatially modulated phases are also widely seen in nature. The order parameters for these phases are associated with nonzero momentum and spontaneously break some or all of the translation invariance. Common examples in condensed matter include spin density waves and charge density waves, while QCD at high baryonic density is anticipated to be in a chiral wave state [4]. Spatially modulated phases that are also superconducting are possible [5] and such FFLO phases have been argued to be realized in several systems [6]. Of particular interest here are p-wave superconducting phases with a helical order. In these phases, the $l = 1$ order parameter points in a given direction in a plane that then rotates as one moves along the direction orthogonal to the plane. They have been discussed, for example, in the context of noncentrosymmetric heavy fermion compounds [7].

Holographic studies of spatially modulated phases were initiated in [8]. The purpose of this Letter is to present the very first construction of fully backreacted black hole solutions that are holographically dual to spatially modulated phases, which are, moreover, superconducting. We will consider a class of gravitational models in $D = 5$ that couple a metric with a gauge field and a two-form potential, which have been shown, using a linearized analysis, to admit black brane solutions that are dual to p-wave superfluid phases with a helical order in $d = 4$ [9].

Our construction of the new black holes allows us to show that the helical p-wave superconducting phase is thermodynamically preferred and that the phase transition is generically second order. The helical order is fixed by wave number k, or equivalently a pitch $p = 2\pi/k$. We calculate $k(T)$ and find that it monotonically decreases down to a finite value as $T \to 0$. We find that the solutions have a vanishing entropy density as $T \to 0$. In this limit they approach smooth domain wall solutions, which we also construct, that interpolate between AdS$_5$ in the UV and a new spatially homogeneous but nonisotropic ground state in the IR, of a type that was recently discussed in [10].

There are other contexts in which spatially modulated black holes should exist, but in general, the construction will require solving nonlinear partial differential equations. By contrast, a key point here is that the black holes for the helical p-wave superconductors can be obtained by solving ordinary differential equations since they are static and also have a Bianchi VII$_0$ symmetry.

The $D = 5$ model.—As in [9], we consider a $D = 5$ model coupling a metric to a gauge field A and a complex two-form C with action

$$S = \int d^5x\sqrt{-g}\Big[R + 12 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}C_{\mu\nu}\bar{C}^{\mu\nu} + \frac{i}{24m}\epsilon^{\mu\nu\rho\sigma\delta}C_{\mu\nu}\bar{H}_{\rho\sigma\delta}\Big], \qquad (1)$$

where a bar denotes complex conjugation and the field strengths, using a form notation, are given by

$$F = dA, \qquad H = dC + ieA \wedge C. \qquad (2)$$

This simple class of models, specified by the parameters m and e, is rather natural. The equations of motion admit a unit radius AdS$_5$ solution with $A = C = 0$, which is dual to some putative conformal field theory (CFT). The massless gauge field A is dual to a current in the CFT, corresponding to a global Abelian symmetry, with scaling dimension

$\Delta = 3$. The two-form C satisfies a first-order equation of motion and is dual to a self-dual rank two tensor operator with $\Delta = 2 + |m|$. In particular, this charged operator has $l = 1$ and thus provides an order parameter for p-wave superconductivity. Such two-forms are common in Kaluza-Klein reductions from $D = 10$ or $D = 11$ supergravity. For example, when $e = 1/\sqrt{3}$, $m = 1$ precisely, this model can be obtained as a consistent Kaluza-Klein truncation of type IIB supergravity on S^5 and moreover, the operator in $N = 4$ SYM dual to C is known [3].

We will study the CFT dual to the AdS_5 vacuum at finite temperature T and chemical potential μ with respect to the global Abelian symmetry by constructing electrically charged asymptotically AdS_5 black branes.

Black hole solutions.—The ansatz for all the black hole solutions that we consider is given by

$$ds^2 = -gf^2dt^2 + g^{-1}dr^2 + h^2\omega_1^2 + r^2(e^{2\alpha}\omega_2^2 + e^{-2\alpha}\omega_3^2),$$
$$C = (ic_1dt + c_2dr)\wedge\omega_2 + c_3\omega_1\wedge\omega_3, \quad A = adt, \tag{3}$$

where the one-forms ω_i are given by

$$\omega_1 = dx_1, \qquad \omega_2 = \cos(kx_1)dx_2 - \sin(kx_1)dx_3,$$
$$\omega_3 = \sin(kx_1)dx_2 + \cos(kx_1)dx_3, \tag{4}$$

and f, g, h, α, $c_{i,}$ and a are all functions of the radial coordinate r only, and k is a constant. Observe that k can be scaled out of the ansatz by scaling h but it has been included for later convenience. The unit radius AdS_5 vacuum solution can be obtained by setting $g = r^2$, $f = 1$, $h = r$, and $\alpha = a = c_i = 0$. Notice that the ansatz (3) is static, and in addition, the constant t and r slices are spatially homogenous of, generically, Bianchi type VII_0.

Substituting this ansatz into the $D = 5$ equations of motion, we find that we can solve for c_1 and c_2:

$$c_1 = -\frac{e^{2\alpha}}{e^{4\alpha}k^2 + m^2h^2}(e^{2\alpha}keac_3 + mhfgc_3'),$$
$$c_2 = \frac{1}{fg}\frac{e^{2\alpha}}{e^{4\alpha}k^2 + m^2h^2}(meahc_3 - e^{2\alpha}kfgc_3'). \tag{5}$$

The remaining equations can be obtained from a one-dimensional action obtained by substituting the ansatz into the action associated with (1).

The equations of motion admit the electrically charged anti–de Sitter-Reissner-Nordström (AdS-RN) black brane solution with $\alpha = c_i = 0$, $f = 1$, $h = r$, and

$$g = r^2 - \frac{r_+^4}{r^2} + \frac{\mu^2}{3}\left(\frac{r_+^4}{r^4} - \frac{r_+^2}{r^2}\right), \qquad a = \mu\left(1 - \frac{r_+^2}{r^2}\right). \tag{6}$$

This solution approaches the unit radius AdS_5 solution as $r \to \infty$. The event horizon is located at $r = r_+$ and the temperature is $T = (6r_+^2 - \mu^2)/6\pi r_+$. This solution is dual to the CFT at finite chemical potential μ and high temperatures. Clearly this phase is spatially homogeneous and isotropic. It was shown in [9] that below a critical temperature, depending on the parameters m, e, and the scale set by μ, this black hole is unstable to the formation of black holes that are dual to p-wave superconductors with helical order.

Let us first discuss the boundary conditions to be imposed for the new black brane solutions. Regularity at the horizon demands that $g(r_+) = a(r_+) = 0$. We then find that the solution at the horizon is specified by six parameters: r_+, $f(r_+)$, $h(r_+)$, $\alpha(r_+)$, $a'(r_+)$, and $c_3(r_+)$. As $r \to \infty$ we approach AdS_5 with asymptotic expansion

$$g = r^2(1 - Mr^{-4} + \cdots), \qquad f = f_0(1 - c_hr^{-4} + \cdots),$$
$$h = r(1 + c_hr^{-4} + \cdots), \qquad \alpha = c_\alpha r^{-4} + \cdots,$$
$$a = f_0(\mu + qr^{-2} + \cdots), \qquad c_3 = c_vr^{-|m|} + \cdots, \tag{7}$$

which is specified by eight parameters M, f_0, c_h, c_α, μ, q, c_v, and k. A number of comments are in order. First, the fact that $h \sim r$ implies that the wave number k can no longer be scaled away. Second, the fall off of c_3 is chosen so that the charged operator dual to the two-form C has no deformation but can spontaneously acquire an expectation value proportional to c_v and spatially modulated in the x_1 direction with period $2\pi/k$. The holographic interpretation of the other UV parameters will be given below. Third, there are two scaling symmetries of the differential equations that allow us to set $\mu = f_0 = 1$, and we will do so later (it is helpful to have them to discuss the thermodynamics). Finally, we have four second order differential equations for h, α, a, c_3, and two first order equations for g, f, and hence a solution is specified by 10 integrations constants. On the other hand, we have 14 parameters in the boundary conditions minus two for the scaling symmetries. We thus expect a two-parameter family of black hole solutions that can be specified by temperature T and wave number k.

Action and thermodynamics.—We analytically continue by setting $t = -i\tau$ and defining $I = -iS$. The total action, including relevant boundary terms, is given by

$$I_{\mathrm{Tot}} = I + \int d\tau d^3x\sqrt{g_\infty}[-2K + 6 + \cdots], \tag{8}$$

with $g_\infty = \lim_{r\to\infty} g^{1/2}fhr^2$ and K is the trace of the extrinsic curvature of the boundary at $r \to \infty$. In (8), the ellipses refer to terms that will not contribute to the class of solutions that we are considering. The period of our Euclidean time is taken to be $\Delta\tau$ and the temperature is then given by $T = (f_0\Delta\tau)^{-1}$. Regularity at $r = r_+$ implies that $T = \frac{f}{f_0}g'(4\pi)^{-1}|_{r=r_+}$. We next define the thermodynamic potential $W = T[I_{\mathrm{Tot}}]_{\mathrm{OS}} \equiv w\mathrm{vol}_3$, where $[I_{\mathrm{Tot}}]_{\mathrm{OS}}$ is the on-shell action. Following the calculation in [11], we obtain the two equivalent expressions

$$w = -M = \varepsilon + 2\mu q - Ts, \tag{9}$$

where we defined the entropy density $s = 4\pi r_+^2h_+$, $\varepsilon = 3M + 8c_h$, and have set $f_0 = 1$. By calculating the on-shell variation of I_{Tot}, including variations of f_0, we deduce that $w = w(T, \mu)$ and the first law

$$\delta w = -s\delta T + 2q\delta\mu. \tag{10}$$

Using (9) we also have $\delta\epsilon = T\delta s - 2\mu\delta q$. The identification of ε with energy density is confirmed by computing the boundary stress-energy tensor [12], again with $f_0 = 1$,

$$T_{tt} = 3M + 8c_h, \quad T_{x_1x_1} = M + 8c_h,$$
$$T_{x_2x_2} = M + 8c_\alpha \cos(2kx_1),$$
$$T_{x_3x_3} = M - 8c_\alpha \cos(2kx_1), \quad T_{x_2x_3} = -8c_\alpha \sin(2kx_1). \tag{11}$$

Observe that when $c_\alpha \neq 0$, the pressures in the x_2, x_3 plane are spatially modulated as one moves along the x_1 direction as one expects for helical order. Defining the average hydrostatic pressure, $\bar{p}$, as minus the average of the trace of the spatial components, we get $\bar{p} = M + 8c_h/3$. The thermodynamically preferred black hole solutions that we construct will have $c_\alpha \neq 0$ and $c_h = 0$. Using (9) we conclude that this class satisfies the thermodynamic relation $\varepsilon + \bar{p} = Ts - 2\mu q$.

Helical superconducting black holes.—The AdS-RN black brane solution (6) is unstable when $e^2 > m^2/2$ [9]. We will now consider the specific model with $m = 1.7$, $e = 1.88$ (for reasons we explain below), and set $\mu = f_0 = 1$. For high temperatures, we only find the AdS-RN black hole solution. The first new black hole solution appears at $T_c \approx 0.0265$ and for $k = k_c \approx 0.550$. Holding this value of k fixed, we numerically construct these black hole solutions all the way down to very low temperatures. Below T_c, as expected from the linearized analysis of [9], there is a continuum of black hole solutions that appear with different values of k. By again holding k fixed, we can construct each of these black holes too, down to low temperatures.

In Fig. 1, we summarized this new two-parameter family of solutions and displayed their free energy w. All of these solutions have smaller free energy than the AdS-RN black brane solutions at the same temperature. At a given temperature $T < T_c$ there is a one parameter family of black hole solutions specified by k and the one with the smallest free energy is depicted by a point on the red line in Fig. 1. Thus, the one-parameter family of solutions specified by the red line characterizes the thermodynamically preferred solutions. Notice that we have a second order phase transition at $T = T_c$, $k = k_c$ and that as the temperature is lowered, the system smoothly moves between black hole solutions with different values of k, all the way down to a very low temperature where $k \equiv k_0 \approx 0.256$. In particular, the $T = 0$ ground state remains spatially modulated.

Interestingly, while the general two-parameter family of solutions have $c_h \neq 0$, the solutions on the red line have (up to numerical accuracy) $c_h = 0$. In Fig. 2, we have plotted, for the red line of solutions, the behavior of c_v and wave number k, which together characterize the helical superconducting order, versus T. Near T_c we find the mean field behavior $c_v \approx 1.7 \times 10^5 T_c^{3.7}(1 - T/T_c)^{1/2}$.

New ground states at $T = 0$.—We are particularly interested in the $T = 0$ limit of the thermodynamically preferred black hole solutions (the red line in Fig. 1), which

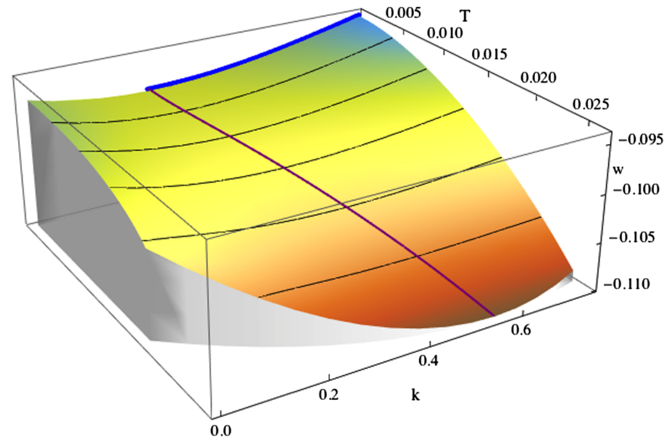

FIG. 1 (color online). The two-parameter family of helical superconducting black holes. The red line (diagonal on the surface) denotes the thermodynamically preferred locus. The blue line (at $T = 0$) is the free energy of some domain wall solutions. Lines of constant T are also marked with black lines.

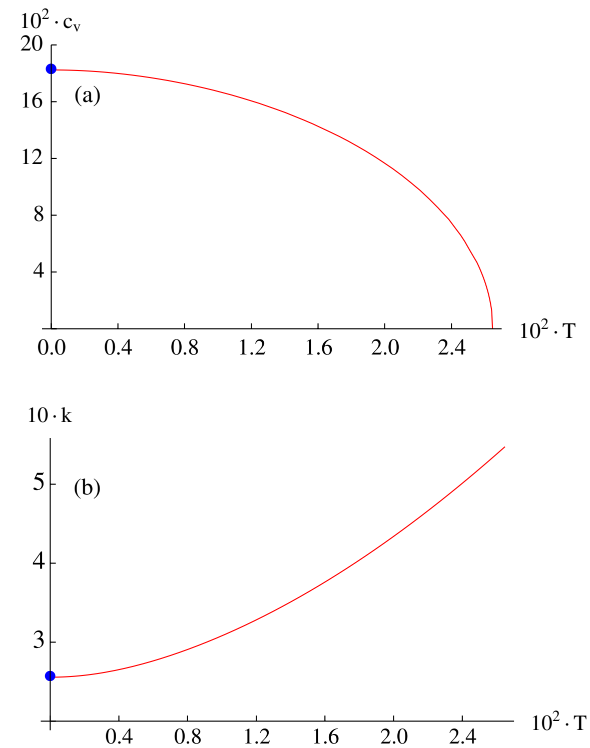

FIG. 2 (color online). Plots of c_v and wave number k, which together fix the helical superconducting order, versus T for the thermodynamically preferred black hole solutions on the red line in Fig. 1. The blue dots depict the quantities for the domain wall solutions. Note the scaled axes.

has $k = k_0 \approx 0.256$. It is helpful to first consider the $T = 0$ limit of the whole class of black holes for general values of k. We find that they all approach a smooth domain wall solution that interpolates between AdS_5 in the UV and a new fixed point in the IR with a scaling symmetry of a type that is very similar to those of [10]. Indeed we checked this explicitly for the range $0.253 \leq k \leq 0.75$ (going to smaller values of k becomes increasingly difficult numerically).

To obtain this new fixed point solution, in (3) we put

$$g = Lr^2, \quad f = \bar{f}_0 r^{z-1}, \quad h = kh_0, \quad \alpha = \alpha_0,$$
$$a = a_0 r^z, \quad c_3 = kc_0 r, \tag{12}$$

where L, h_0, α_0, a_0, c_0, and z are all constant. By scaling t and x_1 we can set $\bar{f}_0 = k = 1$. Notice that this ansatz corresponds to a solution invariant under the anisotropic scaling $r \to \lambda^{-1} r$, $t \to \lambda^z t$, $x_{2,3} \to \lambda x_{2,3}$, and $x_1 \to x_1$. After substituting into the equations of motion, we obtain a system of algebraic equations that can be solved. For the specific case of $m = 1.7$, $e = 1.88$ we find

$$z \approx 1.65\ldots, \qquad L \approx 0.995\ldots, \qquad h_0 \approx 0.993\ldots,$$
$$\alpha_0 \approx -0.380\ldots, \qquad a_0 \approx 0.265\ldots,$$
$$c_0 \approx 3.69\ldots. \tag{13}$$

We next construct domain wall solutions that interpolate between this fixed point in the IR and AdS_5 in the UV [13]. In the UV, we continue to demand the expansion given by (7). To obtain the IR expansion, we first consider perturbations about (12) of the form

$$g = r^2(L + \lambda w_1 r^\delta), \quad f = \bar{f}_0 r^{z-1}(1 + \lambda w_2 r^\delta),$$
$$h = k(h_0 + \lambda w_3 r^\delta), \quad \alpha = \alpha_0 + \lambda w_4 r^\delta,$$
$$a = f_0 a_0 r^z (1 + \lambda w_5 r^\delta), \quad c_3 = kc_0 r(1 + \lambda w_6 r^\delta). \tag{14}$$

After expanding the equations of motion at first order in λ, we obtain a homogeneous linear system of equations of the form $\mathbf{E} \cdot \mathbf{w} = 0$ where $\mathbf{E}$ is a 6×6 matrix that depends on δ. Demanding nontrivial solutions for $\mathbf{w}$, we determine the values of δ by solving the polynomial equation $|\mathbf{E}| = 0$. For the special case $m = 1.7$, $e = 1.88$ the modes with non-negative real parts have $\delta_0 = 0$, $\delta_1 \approx 0.394$, $\delta_2 \approx 0.826$, $\delta_3 \approx 0.847$, and $\delta_4 \approx 2.289$ [14]. The IR expansion is then specified by a constant for each of these modes. Notice that the mode with $\delta_0 = 0$ corresponds to the constant $\bar{f}_0$. This leads to five (real) parameters in the IR. With the eight parameters mentioned for the UV discussed earlier, we now deduce that the domain wall solutions will be specified by a single parameter that we can take to be the wave number k.

We have constructed these solutions for the range $0.253 \leq k \leq 0.75$, which required utilizing high precision numerics. As noted above, we find that the $T = 0$ limit of the black hole solutions approach these domain wall solutions. For example, in Fig. 1, the blue line depicts the free energy of the domain wall solutions for various values of k, showing precise agreement with the $T = 0$ limit of the black holes. Similarly, in Fig. 2, for the domain wall with $k = k_0 \approx 0.256$ we have shown the UV values of c_v and k with a blue dot, and again, we see precise agreement with the corresponding black hole solution.

Final comments.—Let us summarize the main physical results. The $d = 4$ CFT, dual to the AdS_5 solution of our model (1), held at finite chemical potential undergoes a second order phase transition at a critical temperature T_c. The new phase is a helical superconducting phase that spontaneously breaks both the global Abelian symmetry and the three-dimensional spatial Euclidean symmetry down to Bianchi VII_0 symmetry. At T_c the spatial modulation is fixed by a wave number k_c and as the temperature is cooled, the wave number monotonically decreases. The $T = 0$ ground state of the system maintains the helical order with nonvanishing wave number and has an emergent scaling symmetry in the far IR. These homogeneous, non-isotropic ground states at $T = 0$ are holographically described by smooth domain wall solutions. It is natural to next use the AdS/CFT correspondence to analyze transport. Based on the rich optical properties of other helical orders (such as the chiral nematic phase of liquid crystals recently discussed in [16]) we expect interesting results.

We expect analogous solutions for the helical superconductors of the $D = 5$ models with $SU(2) \times U(1)$ gauge fields studied in [9]. In particular, we have checked that the (top-down) Romans theory admits a Bianchi VII_0 ground state solution with scaling exponent $z \approx 3.98$ that we conjecture to be the IR limit of a $T = 0$ domain wall solution. More generally, we now anticipate many other constructions of black holes in $D = 5$, 4 that are dual to other spatially modulated phases, both superconducting and otherwise.

We thank S. Gubser, D. Waldram, and T. Wiseman for helpful discussions. A. D. is supported by EPSRC. J. P. G. is supported by the Royal Society.

[1] S. S. Gubser, Phys. Rev. D 78, 065034 (2008); S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, J. High Energy Phys. 12 (2008) 015.
[2] S. S. Gubser, Phys. Rev. Lett. **101**, 191601 (2008).
[3] F. Aprile, D. Rodriguez-Gomez, and J. G. Russo, J. High Energy Phys. 01 (2011) 056.
[4] D. V. Deryagin, D. Y. Grigoriev, and V. A. Rubakov, Int. J. Mod. Phys. A **7**, 659 (1992).
[5] P. Fulde and R. A. Ferrell, Phys. Rev. **135**, A550 (1964); A. I. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **47**, 1136 (1964) [Sov. Phys. JETP **20**, 762 (1965)].
[6] Y. Matsuda and H. Shimahara, J. Phys. Soc. Jpn. **76**, 051005 (2007).
[7] Y. Yanase and M. Sigrist, J. Phys. Soc. Jpn. **77**, 342 (2008).

[8] S. K. Domokos and J. A. Harvey, Phys. Rev. Lett. **99**, 141602 (2007); S. Nakamura, H. Ooguri, and C.-S. Park, Phys. Rev. D **81**, 004018 (2010).

[9] A. Donos and J. P. Gauntlett, J. High Energy Phys. 12 (2011) 091.

[10] N. Iizuka, S. Kachru, N. Kundu, P. Narayan, N. Sircar, and S. P. Trivedi, arXiv:1201.4861.

[11] J. P. Gauntlett, J. Sonner, and T. Wiseman, J. High Energy Phys. 02 (2010) 060.

[12] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).

[13] Domain walls were constructed for a different $D = 5$ model in [10]. However, unlike our solutions, the breaking of the spatial Euclidean symmetry to Bianchi VII_0 symmetry is not spontaneous.

[14] Obtaining real scaling dimensions was the main reason for focusing on $m = 1.7$, $e = 1.88$. For many values of m, e there will be complex scaling dimensions analogous to [15]. An example is $m = 2$, $e = 2$, and for this case we have also constructed all of the black holes and domain wall solutions, with significantly easier numerics, finding a very similar story with the $T = 0$ ground state matching onto a scaling solution with $z \approx 1.99$.

[15] S. S. Gubser and A. Nellore, Phys. Rev. D **80**, 105007 (2009).

[16] G. W. Gibbons and C. M. Warnick, Phys. Rev. E **84**, 031709 (2011).