

## Quantum Isoperiodic Stable Structures and Directed Transport

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It has been recently found that the so-called isoperiodic stable structures (ISSs) have a fundamental role in the classical current behavior of dissipative ratchets [Phys. Rev. Lett. **106**, 234101 (2011)]. Here I analyze their quantum counterparts, the quantum ISSs (QISSs), which have a fundamental role in the quantum current behavior. QISSs have the simple attractor shape of those ISSs which settle down in short times. However, in the majority of the cases they are strongly different from the ISSs, looking approximately the same as the quantum chaotic attractors that are at their vicinity in parameter space. By adding thermal fluctuations of the size of  $\hbar_{\text{eff}}$  to the ISSs I am able to obtain very good approximations to the QISSs. I conjecture that in general, quantum chaotic attractors could be well approximated by means of just the classical information of a neighboring ISS plus thermal fluctuations. I expect to find this behavior in quantum dissipative systems in general.

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During the last years there have been great advances in the area of directed transport [1–3]. Understood as transport phenomena in periodic systems out of equilibrium, this field has attracted great attention giving rise to many investigations of interdisciplinary nature. In fact, ratchet models have found application in biology [4], nanotechnology [5], and chemistry [6], just to name a few examples. In particular, cold atoms in optical lattices have shown very successful theoretical developments and implementations [7,8]. Moreover, Bose-Einstein condensates have been transported by means of the so-called purely quantum ratchet accelerators [9], where the current has no classical counterpart [10] and the energy grows ballistically [11,12].

The current generation mechanism consists of breaking all spatiotemporal symmetries leading to momentum inversion [13]. Classical deterministic ratchets with dissipation are generally associated with an asymmetric chaotic attractor [14]. Quantum effects were considered to analyze the first so-called quantum ratchets in [15], while a dissipative quantum ratchet, interesting for cold atoms experiments has been introduced in [16]. Very recently, the parameter space of the classical counterpart of this system has been studied in detail [17]. There it has been found that not only chaotic domains but more importantly, a family of isoperiodic stable structures (ISSs) has a fundamental role in understanding the current behavior, a major issue in any ratchet system. Moreover, a complete characterization of this family has been given and a connection with the current values has been also provided. These ISSs are Lyapunov stable islands of different periods and associated currents that are grouped into structures in parameter space and that can be found in generic dynamical systems.

Then, it is natural to ask how this family of ISSs translates into the quantum domain. In this letter I answer this question. I have found that the classical decay time towards these stable structures is a determining factor for the shape of the

corresponding quantum ISSs (QISSs). Quantum ISSs are defined as the quantum manifestations of the classical ISSs. More precisely, I refer to QISSs regardless of them being of a simple or a chaotic (complex) shape. This is in clear contrast with classical ISSs, which are simple attractors in our case. In fact, the majority of these classical structures have very long transient times making the QISSs look like the quantum chaotic attractors at their vicinity in parameter space, whose classical counterparts are already chaotic (this being the definition of a quantum (chaotic) attractor). In comparatively few cases I have found quantum structures similar to these classically simple objects (periodic points in the case of maps). The simple shape of the ISSs seems to be an almost exclusive property of the classical system. On the other hand, by adding thermal fluctuations of the size of  $\hbar_{\text{eff}}$  to the classical dynamics very good approximations to the QISSs were obtained.

I investigate a paradigmatic dissipative ratchet system given by the map [16,17]

$$\bar{n} = \gamma n + k[\sin(x) + a \sin(2x + \phi)], \quad \bar{x} = x + \tau \bar{n}, \quad (1)$$

where  $n$  is the momentum variable conjugated to  $x$ ,  $\tau$  is the period of the map and  $\gamma$  is the dissipation parameter. This dynamics can be interpreted as that of a particle moving in one dimension [ $x \in (-\infty, +\infty)$ ] in a periodic kicked asymmetric potential:

$$V(x, t) = k \left[ \cos(x) + \frac{a}{2} \cos(2x + \phi) \right] \sum_{m=-\infty}^{+\infty} \delta(t - m\tau), \quad (2)$$

where  $\tau$  is the kicking period, and subject to a dissipation  $0 \leq \gamma \leq 1$ . When  $\gamma = 0$  the particle is overdamped and for  $\gamma = 1$  we recover the conservative evolution. As usual, the directed transport appears due to broken spatial ( $a \neq 0$  and  $\phi \neq m\pi$ ) and temporal ( $\gamma \neq 1$ ) symmetries. It is useful to notice that the classical dynamics only depends

on the parameter  $K = k\tau$  by means of introducing the rescaled momentum  $p = \tau n$ .

In order to quantize this model I follow the standard procedure:  $x \rightarrow \hat{x}$ ,  $n \rightarrow \hat{n} = -i(d/dx)$  ( $\hbar = 1$ ). Since  $[\hat{x}, \hat{p}] = i\tau$ , the effective Planck constant is  $\hbar_{\text{eff}} = \tau$ . The classical limit corresponds to  $\hbar_{\text{eff}} \rightarrow 0$ , while  $K = \hbar_{\text{eff}} k$  remains constant. The final ingredient, the dissipation, is introduced by means of the master equation [18] for the density operator  $\hat{\rho}$  of the system

$$\dot{\hat{\rho}} = -i[\hat{H}_s, \hat{\rho}] - \frac{1}{2} \sum_{\mu=1}^2 \{\hat{L}_{\mu}^{\dagger} \hat{L}_{\mu}, \hat{\rho}\} + \sum_{\mu=1}^2 \hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{\dagger}. \quad (3)$$

Here  $\hat{H}_s = \hat{n}^2/2 + V(\hat{x}, t)$  is the system Hamiltonian,  $\{\cdot, \cdot\}$  is the anticommutator, and  $\hat{L}_{\mu}$  are the Lindblad operators given by

$$\begin{aligned} \hat{L}_1 &= g \sum_n \sqrt{n+1} |n\rangle\langle n+1|, \\ \hat{L}_2 &= g \sum_n \sqrt{n+1} |-n\rangle\langle -n-1|, \end{aligned} \quad (4)$$

with  $n = 0, 1, \dots$  and  $g = \sqrt{-\ln\gamma}$  (due to the Ehrenfest theorem). This is a standard choice of Lindblad operators for introducing dissipation (see [19] and references therein), but other choices are possible. In all cases I have evolved  $10^6$  classical random initial conditions having  $p \in [-\pi, \pi]$  and  $x \in [0, 2\pi]$  ( $\langle p_0 \rangle = 0$ ), and their quantum counterpart.

In [17] the important role that ISSs have on the ratchet currents corresponding to dissipative systems has been shown. There, three main kinds were identified and called  $B_M$ ,  $C_M$  and  $D_M$ , where  $M$  stands for an integer or rational number and corresponds to the mean momentum of these structures in units of  $2\pi$ . With perhaps the only exception of  $\gamma \rightarrow 1$  (i.e., near the conservative limit), ISSs organize the parameter space structure and then are essential to understand the current behavior. Though I have checked the general validity of the results by sampling several points in the parameter space, I have selected a representative case for each kind of these structures. Also, I have studied a chaotic attractor case in the immediate vicinity of one of them in parameter space, for comparison purposes. It is important to underline that all the chaotic attractors in the immediate vicinity of a given ISS (with different  $\gamma$ ,  $k$ , or both) are of similar shape in general. In Fig. 1 I show the phase space portraits (after 50 time steps of the map) corresponding to the chosen cases, i.e.,  $B_1$  [ $\gamma = 0.2$ ,  $k = 8.2$ , Figs. 1(a) and 1(b)],  $C_{-1}$  [ $\gamma = 0.64$ ,  $k = 5.6$ , Figs. 1(c) and 1(d)], and  $D_{-1}$  [ $\gamma = 0.29$ ,  $k = 11.9$ , Figs. 1(e) and 1(f)]. Only for the first case I have found a pointlike structure (besides the uncertainty restrictions) similar to the classical simple attractors marked with black dots in Figs. 1(a), 1(c), and 1(e). The two other kinds (and even other regions of the same  $B_1$  large structure in parameter space, and also other  $B_M$  with  $M \neq 1$ ) behave like a quantum chaotic attractor. This strikingly different

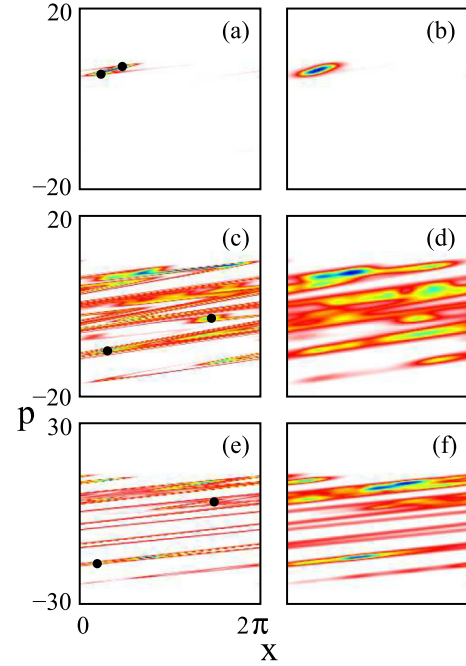


FIG. 1 (color online). Phase space distributions corresponding to the ISSs labeled by  $B_1$  ( $\gamma = 0.2$ ,  $k = 8.2$ , first row),  $C_{-1}$  ( $\gamma = 0.64$ ,  $k = 5.6$ , middle row), and  $D_{-1}$  ( $\gamma = 0.29$ ,  $k = 11.9$ , bottom row) in [17]. Lower to higher values of these distributions go from white to (rainbow colors) grays. Left column shows the classical Poincaré maps while the right column shows the Husimi functions. In (a), (c), and (e) the periodic points of each ISS are marked by means of a black dot. The surrounding distributions correspond to the ISSs at finite  $T$  (for (a)  $T = 0.058$ , for (c)  $T = 0.12$ , and for (e)  $T = 0.07$ ). In (b), (d), and (f)  $\hbar_{\text{eff}} \approx 0.082$ . Notice that for (e) and (f)  $p \in [-30, 30]$ , while for the rest  $p \in [-20, 20]$ .

behavior is due to the effects of the noise induced by quantization [19,20]. The way to prove this is to introduce fluctuations of the size of  $\hbar_{\text{eff}}$  in the classical model so as to induce classical chaos. I do this by changing  $\bar{n}' \rightarrow \bar{n}$  in Eq. (1), where  $\bar{n}' = \bar{n} + \xi$ . The thermal noise  $\xi$  is related to  $\gamma$ , according to  $\langle \xi^2 \rangle = 2(1 - \gamma)k_B T$ , where  $k_B$  is the Boltzmann constant (which I take equal to 1) and  $T$  is the temperature. In this expression I have chosen the factor  $(1 - \gamma)$  in order not to have noise in the conservative limit, however other choices are possible. Finally I take  $T \approx \hbar_{\text{eff}}/[2(1 - \gamma)]$ . As a result, I obtain the classical distributions that surround the periodic points and that are shown in Figs. 1(a), 1(c), and 1(e). They are remarkably similar to the corresponding quantum chaotic attractors. Even in the first case [i.e., Figs. 1(a) and 1(b) corresponding to  $B_1$ ], adding fluctuations is sufficient to reproduce the quantum simple attractor (which is not pointlike, of course). This is confirmed by means of the overlap measure defined as  $O = \iint \mathcal{L}(x, p) \mathcal{H}(x, p) dx dp$  ( $\mathcal{L}(x, p)$  and  $\mathcal{H}(x, p)$  are normalized phase space distributions with the same discretization) which compares the whole distributions, not just the first moment. In this case these distributions correspond to the

classical Liouville distribution and the quantum Husimi function. For  $B_1$  and  $T = 0$  I obtain  $O \approx 0.1$ , while for  $T = 0.058$  the result is  $O \approx 0.87$ , the other two cases behave the same way, resulting in  $O \approx 0.07$  and  $O \approx 0.91$  for  $T = 0$  and  $T = 0.12$  respectively in the  $C_{-1}$  case, and  $O \approx 0.14$  and  $O \approx 0.87$  for  $T = 0$  and  $T = 0.07$  respectively in the  $D_{-1}$  ISS.

I have found that the convergence to a quantum simple attractor is not uniform with respect to  $\hbar_{\text{eff}}$ , but highly dependent on the ISS in question. To illustrate this result I show the QISS corresponding to the  $B_1$  case for  $\hbar_{\text{eff}} \approx 0.246$  in Fig. 2(a). It is clear that for this higher value of  $\hbar_{\text{eff}}$  the QISS starts to resemble the neighboring chaotic attractor. On the contrary, when I study the QISS corresponding to  $D_{-1}$  for  $\hbar_{\text{eff}} \approx 0.027$  [see Fig. 2(b)] the chaotic nature of the quantum attractor shows no sign of vanishing. On the other hand, it is also very important to point out that the QISS have great influence on the quantum chaotic attractors surrounding them. In fact, I have taken  $\gamma = 0.26$  and  $k = 11.9$  in the chaotic area  $A$  (according to [17] nomenclature), in the vicinity of the “shrimp” structure  $D_{-1}$ . Here the classical attractor resembles the quantum one without the need of additional fluctuations, giving an overlap  $O \approx 0.73$  at  $T = 0$ . In this case the chaotic mixing is enough to recover the quantum chaotic shape. But, most importantly, the quantum chaotic attractor is very similar to the corresponding QISS, their overlap [between the distributions shown in Figs. 1(f) and 2(d)] being  $O \approx 0.99$ . This suggests the idea that general quantum chaotic attractors could be acceptably reproduced with just the classical information corresponding to a neighboring ISS plus thermal fluctuations. This gives to the QISSs an even more relevant role in the understanding of quantum dissipative ratchets and also of quantum dissipative systems in general.

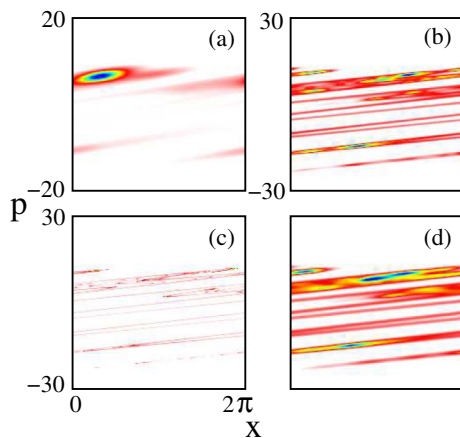


FIG. 2 (color online). In (a) I show the Husimi function for the  $B_1$  QISS [see Figs. 1(a) and 1(b)] for  $\hbar_{\text{eff}} \approx 0.246$ . The same (color) gray scale as in Fig. 1 is used. In (b) the  $D_{-1}$  QISS for  $\hbar_{\text{eff}} \approx 0.027$  is shown. In (c) and (d) the classical ( $T = 0$ ) and quantum ( $\hbar_{\text{eff}} \approx 0.082$ ) chaotic attractors, respectively, neighboring  $D_{-1}$ , for which  $\gamma = 0.26$  and  $k = 11.9$ .

The influence of QISSs and the effect of thermal noise in the ratchet current  $J = \langle p \rangle$  (where  $\langle \rangle$  stands for either the classical or quantum averages) can be seen in Figs. 3 and 4, where I compare the classical and quantum  $J$  for the cases of Fig. 1 and the chaotic attractor of Figs. 2(c) and 2(d). The upper lines of Fig. 3 represent the  $B_1$  case. The dot-dashed black curve corresponds to the classical current at  $T = 0$ , which stabilizes in a relatively short time (50 time steps) when compared to the same curve for the  $C_{-1}$  case in the same figure (in fact, this latter one stabilizes in more than 250 time steps). In general, the ISSs have long transients, settling down in times 1 order of magnitude longer than this. This seems to be the main reason for the  $B_1$  case under scrutiny to show a quantum simple behavior in contrast with the other cases. I am currently investigating the details of this behavior [21].

In this Figure, the (green) gray full lines correspond to the quantum current, while the black dashed thin lines correspond to the classical  $J$  at finite  $T$ . The effects of the thermal environment quickly stabilize this current. This is specially evident in the  $C_{-1}$  case, which clearly loses even the period 2 bumps exhibited by the lower dot-dashed black line. Moreover, in both cases displayed in Fig. 3 the agreement of the classical  $J$  at finite  $T$  with the quantum current is excellent, reflecting what I have already shown by means of the phase space distributions.

In Fig. 4 I make the same comparison as in Fig. 3, but in this case for the QISS corresponding to the  $D_{-1}$  kind with the chaotic attractor in its vicinity. Again, the classical current for the ISS stabilizes very slowly (it takes around 700 time steps). On the other hand, the quantum  $J$  stabilizes very quickly (in 7–8 time steps) as does the classical one at finite  $T$  (their agreement being excellent). We can see that the classical current corresponding to the chaotic attractor case also stabilizes very quickly, without the need of additional fluctuations. In the classically chaotic case

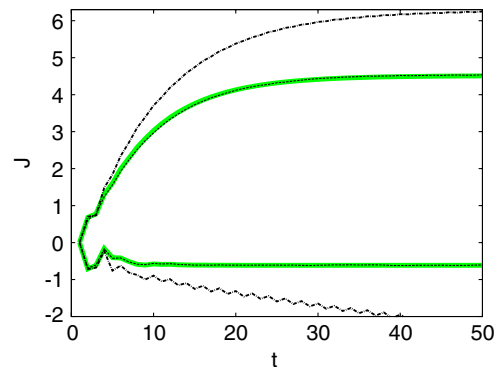


FIG. 3 (color online). Classical and quantum current  $J$  as a function of map time steps  $t$ . Black dot-dashed lines correspond to classical values at  $T = 0$ , black dashed thin lines to classical values at the finite temperatures of Fig. 1, and (green) gray full lines to the quantum values. Upper lines correspond to the  $B_1$  ISS, and lower ones to  $C_{-1}$ .

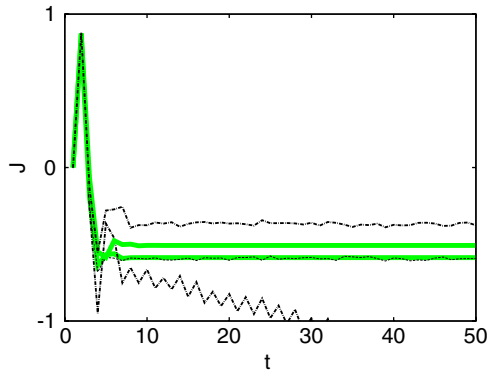


FIG. 4 (color online). Same as in Fig. 3, but for the  $D_{-1}$  ISS (lower lines) and the chaotic attractor in its vicinity shown in Figs. 2(c) and 2(d) (upper lines).

the stabilization of the current and its main features (like the generic low values and lack of periodic fluctuations) are already determined by the chaotic mixing.

In conclusion, I have found that QISSs have a fundamental role in the behavior of the current in quantum dissipative ratchets, as ISSs have in their classical counterparts. This has been unveiled by means of analyzing a paradigmatic system in the ratchet and chaos (classical and quantum) literature. QISSs have the simple attractor shape of the classical ISSs only in the few cases where the time in which they stabilize is very short. But in general they have an extended, chaotic attractor shape in phase space. This has very important practical implications for the currents measured with cold atoms in a periodic standing wave of light, systems where our model has direct application (for example, sodium atoms in a laser field with wave length  $\lambda = 589$  nm). The QISSs could be observed by means of state reconstruction techniques [22]. Moreover, I have found that the behavior of the QISSs can be understood by means of just the classical information contained in the corresponding ISSs plus thermal fluctuations of the order of  $\hbar_{\text{eff}}$ . In addition, the quantum chaotic attractors in their vicinity have a very similar structure. This makes us conjecture that in general it should be possible to approximate any quantum chaotic attractor by means of the essential classical information contained in a neighboring ISS (plus stochastic fluctuations). This is expected to be a general result valid for any quantum dissipative system which has a contractive (in phase space) kind of noise and it will be the matter of future investigations [21].

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- [1] R.P. Feynman, *Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. 1.
- [2] P. Reimann, *Phys. Rep.* **361**, 57 (2002).
- [3] S. Kohler, J. Lehmann, and P. Hänggi, *Phys. Rep.* **406**, 379 (2005).
- [4] F. Jülicher, A. Ajdari, and J. Prost, *Rev. Mod. Phys.* **69**, 1269 (1997); G. Mahmud, C. J. Campbell, K. J. M. Bishop, Y. A. Komarova, O. Chaga, S. Soh, S. Huda, K. Kandere-Grzybowska, and B. A. Grzybowski, *Nature Phys.* **5**, 606 (2009); G. Lambert, D. Liao, and R. H. Austin, *Phys. Rev. Lett.* **104**, 168102 (2010).
- [5] R. D. Astumian, *Science* **276**, 917 (1997); D. Reguera, A. Luque, P. S. Burada, G. Schmid, J. M. Rubí, and P. Hänggi, *Phys. Rev. Lett.* **108**, 020604 (2012).
- [6] J. B. Gong and P. Brumer, *Annu. Rev. Phys. Chem.* **56**, 1 (2005); G. G. Carlo, L. Ermann, F. Borondo, and R. M. Benito, *Phys. Rev. E* **83**, 011103 (2011).
- [7] P. H. Jones, M. Goonasekera, D. R. Meacher, T. Jonckheere, and T. S. Monteiro, *Phys. Rev. Lett.* **98**, 073002 (2007); T. Salger, S. Kling, T. Hecking, C. Geckeler, L. Morales-Molina, and M. Weitz, *Science* **326**, 1241 (2009); F. Renzoni, arXiv:1112.0851.
- [8] T. S. Monteiro, P. A. Dando, N. A. C. Hutchings, and M. R. Isherwood, *Phys. Rev. Lett.* **89**, 194102 (2002); G. G. Carlo, G. Benenti, G. Casati, S. Wimberger, O. Morsch, R. Mannella, and E. Arimondo, *Phys. Rev. A* **74**, 033617 (2006).
- [9] M. Sadgrove, M. Horikoshi, T. Sekimura, and K. Nakagawa, *Phys. Rev. Lett.* **99**, 043002 (2007); I. Dana, V. Ramareddy, I. Talukdar, and G. S. Summy, *Phys. Rev. Lett.* **100**, 024103 (2008).
- [10] E. Lundh and M. Wallin, *Phys. Rev. Lett.* **94**, 110603 (2005); D. Poletti, G. G. Carlo, and B. Li, *Phys. Rev. E* **75**, 011102 (2007).
- [11] A. Kenfack, J. Gong, and A. K. Pattanayak, *Phys. Rev. Lett.* **100**, 044104 (2008); J. Wang and J. Gong, *Phys. Rev. E* **78**, 036219 (2008).
- [12] M. Sadgrove, M. Horikoshi, T. Sekimura, and K. Nakagawa, *Eur. Phys. J. D* **45**, 229 (2007).
- [13] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, *Phys. Rev. Lett.* **84**, 2358 (2000).
- [14] J. L. Mateos, *Phys. Rev. Lett.* **84**, 258 (2000).
- [15] P. Reimann, M. Grifoni, and P. Hänggi, *Phys. Rev. Lett.* **79**, 10 (1997).
- [16] G. G. Carlo, G. Benenti, G. Casati, and D. L. Shepelyansky, *Phys. Rev. Lett.* **94**, 164101 (2005).
- [17] A. Celestino, C. Manchein, H. A. Albuquerque, and M. W. Beims, *Phys. Rev. Lett.* **106**, 234101 (2011).
- [18] G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).
- [19] T. Dittrich and R. Graham, *Europhys. Lett.* **7**, 287 (1988).
- [20] A. R. Kolovsky, *Chaos* **6**, 534 (1996).
- [21] G. G. Carlo (to be published).
- [22] M. Bienert, F. Haug, W. P. Schleich, and M. G. Raizen, *Phys. Rev. Lett.* **89**, 050403 (2002).