

# Geometric Heat Flux for Classical Thermal Transport in Interacting Open Systems

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We study classical heat conduction in a dissipative open system composed of interacting oscillators. By exactly solving a twisted Fokker-Planck equation which describes the full counting statistics of heat flux flowing through the system, we identify the geometric-phase-like effect and examine its impact on the classical heat transport. We find that the nonlinear interaction as well as the closely related temperature dependence of system parameters are crucial in manifesting the geometric-phase contribution of heat flux. Finally, we propose an electronic experiment based on *RC* circuits to verify our theoretical predictions.

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Understanding the general features of transports is one of the main goals in nonequilibrium statistical physics. Among many others, time-dependent driven transports like driven particle (mass, probability) transport and driven heat conduction are attracting increasing attention. In particular, the latter is of special interest [1–9] because of both its theoretical and practical importance in phononics [10], where one may utilize temporal modulations to alternatively achieve flexible dynamic control of thermal energy in various phononic devices [10].

In driven quantum systems, an intrinsic geometric contribution of the phase of a wave function will be emergent [11] and it has been proved to have profound effects on many physical properties including thermal related ones [12–14]. Similar geometric contributions have been also uncovered in the driven transport of noninteracting particles [15–17]. These pioneering efforts culminated in the discovery of geometric-phase-like contribution in generating functions [18,19], which then inspired the identification of geometric heat flux in a single quantum junction [9]. Namely, even under slow modulations, heat flux and fluctuations are not merely a simple temporal average of their static counterparts, but contain an extra geometric contribution regardless of driving rates.

Unlike noninteracting particle transport, heat conduction in solid is typically modeled by interacting lattices with reservoirs, where energy is transported in the absence of particle flow [20]. Therefore, whether and how the geometric contribution can emerge in classical heat conduction is still an open question. Moreover, the nonlinearity (anharmonicity) has been found of special importance in phononic devices [10]. However, the role of nonlinear interaction in the manifestation of geometric heat flux is still not yet explored, although works about time-dependent classical heat conduction in interacting lattices have already been carried out [2–7].

In this Letter, we shall address the above mentioned objectives by exactly solving a twisted Fokker-Planck equation, which describes the full counting statistics of heat flux flowing through a classical open system of interacting oscillators. We identify the geometric-phase effect on generating functions and examine its impact on the classical heat transport. In particular, we find that the nonlinearity of interaction as well as the related temperature dependence of system parameters are crucial to the manifestation of geometric heat flux. Otherwise, for a linear system without temperature-dependent parameters, the geometric-phase effect is absent or only observable in high order heat fluctuations. Furthermore, by pointing out the analogy of a coupled *RC* electric circuit and interacting oscillators, we are able to implement an electric experiment to verify the theoretical predictions of geometric-phase effects in heat transport.

We start with a typical interacting open system: two coupled Brownian oscillators in contact with two heat baths, as shown in Fig. 1. The vibrational dynamics is described by a set of Langevin equations:  $m_i \ddot{x}_i = -\partial_{x_i} V(x_1, x_2) - \gamma_i \dot{x}_i + \xi_i$ , ( $i = 1, 2$ ), where  $m_i$  and  $x_i$  are the mass and displacement of oscillator  $i$ .  $V(x_1, x_2)$  denotes the interaction potential.  $\gamma_i$  depicts the viscosity of bath  $i$ , or say, the coupling strength between oscillator  $i$  and bath  $i$ .  $\xi_i$  is the white noise with variance  $\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma_i T_i \delta_{ij} \delta(t - t')$ , where  $T_i$  is the temperature of bath  $i$ .  $Q(t) = \int_0^t \partial_{x_1} V(x_1, x_2) \dot{x}_1 dt'$  is defined to describe the heat

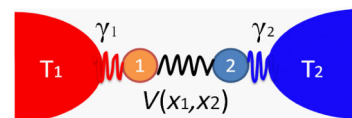


FIG. 1 (color online). A sketch of two coupled classical Brownian oscillators in contact with two Langevin heat baths.

transferred from heat bath 1 to 2 during time  $t$ . This vibrational dynamics is a typical interacting model for heat transport which has been also used to describe the well-known Feynman ratchet-pawl model [21]. For microdynamics at low Reynolds number where oscillators either possess nanosizes or move in an extremely viscous media, the inertia terms are negligible [22], and in turn we have an overdamped dynamics:

$$\gamma_1 \dot{x}_1 + \partial_{x_1} V(x_1, x_2) = \xi_1, \quad \gamma_2 \dot{x}_2 + \partial_{x_2} V(x_1, x_2) = \xi_2. \quad (1)$$

For an harmonic coupling  $V(x_1, x_2) = k(x_1 - x_2)^2/2$  with spring constant  $k$ , the system can be described by a Fokker-Planck equation [23]:

$$\frac{\partial \rho(y, t)}{\partial t} = \left[ \left( \frac{T_1}{\gamma_1} + \frac{T_2}{\gamma_2} \right) \frac{\partial^2}{\partial y^2} + \left( \frac{k}{\gamma_1} + \frac{k}{\gamma_2} \right) \frac{\partial}{\partial y} y \right] \rho(y, t), \quad (2)$$

where  $y(t) = x_1(t) - x_2(t)$ . Then the transferred heat from bath 1 to 2 within time  $t$  is  $Q(t) = \int_0^t dt' ky(\xi_1 - ky)/\gamma_1$ .

To study the time-dependent driven heat transport, we introduce the characteristic function of the joint probability  $\rho(y, Q, t)$ , defined as  $z(y, \chi, t) = \int_{-\infty}^{\infty} dQ e^{i\chi Q} \rho(y, Q, t)$ . This characteristic function satisfies a twisted Fokker-Planck equation:

$$\partial_t z(y, \chi, t) = L_\chi(t) z(y, \chi, t), \quad (3)$$

where, following derivations from [24,25], one can find,

$$L_\chi = \left( \frac{T_1}{\gamma_1} + \frac{T_2}{\gamma_2} \right) \frac{\partial^2}{\partial y^2} + k \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} - \frac{2T_1}{\gamma_1} i\chi \right) y \frac{\partial}{\partial y} + k^2 \left( \frac{T_1}{\gamma_1} (i\chi)^2 - \frac{1}{\gamma_1} i\chi \right) y^2 + k \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} - \frac{T_1}{\gamma_1} i\chi \right). \quad (4)$$

The twisted Fokker-Planck operator  $L_\chi(t)$  is generally time-dependent, wherein parameters  $k(t)$ ,  $\gamma_j(t)$  and  $T_j(t)$  could be subject to adiabatically cyclic modulations. The ‘‘adiabatic’’ here means the modulation period  $T_p$  is much larger than the system’s characteristic time scale of relaxation,  $T_c$ , namely [26],

$$T_p \gg T_c = \frac{\gamma_1 \gamma_2}{k(\gamma_1 + \gamma_2)}. \quad (5)$$

It is easy to verify that Eqs. (3) and (2) have the same initial condition  $z(y, \chi, 0) = \rho(y, 0)$ . When  $\chi = 0$ , we have  $z(y, 0, t) = \rho(y, t)$  from the definition and in turn Eq. (3) reduces to Eq. (2). Integrating over the degree of freedom  $y$  in  $z(y, \chi, t)$ , we obtain the characteristic function of  $Q$ :  $Z(\chi, t) = \int dy z(y, \chi, t) = \int dQ e^{i\chi Q} P(Q, t)$ , with  $P(Q, t) = \int dy \rho(y, Q, t)$ . Thus, the cumulant generating function is  $G(\chi) = \lim_{t \rightarrow \infty} t^{-1} \ln Z(\chi, t)$ , which generates the  $n$ -order cumulant of heat fluctuations:  $\lim_{t \rightarrow \infty} \langle \langle Q^n \rangle \rangle / t = \partial_{i\chi}^n G(\chi) |_{\chi=0}$ .

Following [18], the cumulant generating function of adiabatically driven system can be separated into two parts—the dynamic contribution and the geometric one:  $G(\chi) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln Z(\chi, t) = G_{\text{dyn}} + G_{\text{geom}}$ . The dynamic

contribution,  $G_{\text{dyn}} = \frac{1}{T_p} \int_0^{T_p} dt \lambda_0(\chi, t)$ , with  $\lambda_0(\chi, t)$  denoting the ground-state eigenvalue of  $L_\chi(t)$ , survives whenever system parameters are static, or experience single or multiple modulations, whereas the appearance of the geometric contribution  $G_{\text{geom}}$  requires at least two parameter modulations. For the case of periodically driving pairs  $(u_1(t), u_2(t))$ , which could be chosen from any two of  $k, \gamma_j, T_j$ , we have [26]

$$G_{\text{geom}} = -\frac{1}{T_p} \iint_{u_1 u_2} du_1 du_2 \mathcal{F}_{u_1 u_2}(\chi), \quad (6)$$

$$\mathcal{F}_{u_1 u_2}(\chi) \equiv \int_{-\infty}^{\infty} dy \left[ \frac{\partial \varphi_0}{\partial u_1} \frac{\partial \psi_0}{\partial u_2} - \frac{\partial \varphi_0}{\partial u_2} \frac{\partial \psi_0}{\partial u_1} \right], \quad (7)$$

where  $\varphi_0(\psi_0)$  denotes the corresponding left(right) ground-state eigenfunction, and the subscript  $u_1 u_2$  denotes the integral area enclosed by the modulating contour. Clearly,  $\mathcal{F}_{u_1 u_2}(\chi)$  has the physical meaning of curvature of the parameter space  $(u_1, u_2)$  for  $L_\chi$ . It is of pure geometric origin and is independent of the modulation speed. Mathematically,  $G_{\text{geom}}$  is an analog of the geometric Berry phase [19] in quantum mechanics, where the wave function will gain an extra phase after a cyclic evolution [27,28]. Similarly, in the full counting statistics of cyclic driven systems, the cumulant generating function (analog of phase) in the exponent of the characteristic function (analog of wave function) will also gain an additional term [18,19]. Both extra terms share the similar geometric origin from the nontrivial curvature in the system’s parameter space. In turn the  $n$ th cumulant of heat fluctuations has two separate contributions as well [9,18]:

$$\lim_{t \rightarrow \infty} \frac{\langle \langle Q^n \rangle \rangle}{t} = \frac{\partial^n G_{\text{dyn}}}{\partial (i\chi)^n} \Big|_{\chi=0} + \frac{\partial^n G_{\text{geom}}}{\partial (i\chi)^n} \Big|_{\chi=0}. \quad (8)$$

Of prime interest is the first cumulant, the average heat flux  $J = J_{\text{dyn}} + J_{\text{geom}}$ , with

$$J_{\text{dyn}} = \frac{1}{T_p} \int_0^{T_p} dt \frac{\partial \lambda_0(\chi, t)}{\partial (i\chi)} \Big|_{\chi=0}, \quad (9)$$

$$J_{\text{geom}} = -\frac{1}{T_p} \iint_{u_1 u_2} du_1 du_2 \frac{\partial \mathcal{F}_{u_1 u_2}(\chi)}{\partial (i\chi)} \Big|_{\chi=0}. \quad (10)$$

Apparently, to study the geometric heat flux as well as other interest transport properties, we need to solve the eigenproblem of  $L_\chi$ , which however spans the infinite-dimensional Hilbert space and is usually difficult to tackle. Fortunately, after some algebra, we can cast Eqs. (3) and (4) to the Schrödinger’s eigenproblem of quantum harmonic oscillator [26], with the help of which we analytically obtain the ground-state eigenvalue for  $L_\chi(t)$ ,

$$\lambda_0(\chi, t) = r(1 - \theta)/2, \quad (11)$$

and the right ground-state eigenfunction,

$$\psi_0(y, \chi, t) = \exp\left(-\frac{r + r\theta - 2i\chi k D_1}{4D} y^2\right), \quad (12)$$

as well as the corresponding left eigenfunction,

$$\varphi_0(y, \chi, t) = \sqrt{\frac{r\theta}{2\pi D}} \exp\left(\frac{r - r\theta - 2i\chi k D_1}{4D} y^2\right), \quad (13)$$

where  $\theta = \sqrt{1 - 4k^2 D_1 D_2 / r^2 (-\chi + i(1/T_2 - 1/T_1))} \chi$ ,  $r = r_1 + r_2$ ,  $D = D_1 + D_2$  with  $r_j = k/\gamma_j$  and  $D_j = T_j/\gamma_j$ .

Substituting Eq. (11) into Eq. (9), we see that the dynamic heat flux is just the temporal average of its static counterpart:  $J_{\text{dyn}} = T_p^{-1} \int_0^{T_p} dt J_{\text{st}}(t)$ , with  $J_{\text{st}} \equiv \partial_{i\chi} \lambda_0|_{\chi=0} = k(T_1 - T_2)/(\gamma_1 + \gamma_2)$ , where the integral concerns all possible time-dependent parameters. It is interesting to notice that generally given  $T_1(t) = T_2(t)$ , not only the average, but all the odd order cumulants of the dynamic flux will vanish as well, due to the even symmetry  $\lambda_0(\chi) \equiv \lambda_0(-\chi + i(1/T_2 - 1/T_1)) = \lambda_0(-\chi)$ .

Substituting Eqs. (12) and (13) into Eq. (7) and in turn Eq. (10), we can then study the geometric heat flux  $J_{\text{geom}}$  under any pair-parameter manipulations. We first consider a special case of two system-bath couplings  $\gamma_1(t)$ ,  $\gamma_2(t)$  being modulated. In this case, we find the curvature  $\mathcal{F}_{\gamma_1 \gamma_2} \equiv 0$ , that is, no matter how arbitrarily one drives these two couplings, the geometric contributions to all cumulants of heat transport are always zero. The geometric effect is absent in the case of merely modulating system-bath couplings. Similar phenomenon is also observed in quantum heat transport [9], which implies that this may have connection with some universal pumping restrictions [29] in open systems. Furthermore, this absence of geometric heat flux may relate to a no-pumping theorem of a different quantity, probability current, in closed driven systems without explicitly connecting with heat baths [30]. It shows that, in terms of our Eq. (2), the probability current is absent when  $\frac{k(1/\gamma_1 + 1/\gamma_2)}{(T_1/\gamma_1 + T_2/\gamma_2)}$  is time independent. In fact, when  $T_1 = T_2$ , this ratio is indeed independent of  $\gamma_i$  so that there is no pumping of probability current no matter how we drive  $(\gamma_1, \gamma_2)$ .

In the case of modulating any other combination of two system parameters, however, the geometric contribution emerges. For example, if we modulate one system-bath

coupling  $\gamma_2(t)$  and one bath temperature  $T_1(t)$ , the nonzero first derivative of curvature,  $\partial_{i\chi} \mathcal{F}_{\gamma_2 T_1}|_{\chi=0} = -\gamma_1 \gamma_2 / [2(\gamma_1 + \gamma_2)^3] \neq 0$  can induce nonzero geometric heat flux  $J_{\text{geom}} \neq 0$ . For another example, if two baths are kept isothermal  $T_1 = T_2 = T_0$ , then the dynamic heat flux will be always zero,  $J_{\text{dyn}} = 0$ . Even so, we still can realize the nonzero heat transport through the geometric contribution by modulating, like  $(k, \gamma_2)$ , so that  $J = J_{\text{geom}} = T_p^{-1} \int_{k, \gamma_2} dk d\gamma_2 \{\gamma_1 T_0 / [2k(\gamma_1 + \gamma_2)^2]\}$ .

The most typical modulation is driving bath temperatures  $(T_2, T_1)$  [3–6], because implementing this protocol is much easier than modulating  $k(t)$  and  $\gamma_j(t)$  in practice. The curvature for this two-temperature driving reads

$$\mathcal{F}_{T_2 T_1} = \frac{\gamma_1 \gamma_2 (\gamma_2 - \gamma_1) (i\chi)^2}{2(\gamma_1 + \gamma_2)^3 \theta^3}. \quad (14)$$

When the couplings are symmetric,  $\gamma_2 = \gamma_1$ , one can see that  $\mathcal{F}_{T_2 T_1} \equiv 0$ . That is, the geometric-phase effect is absent in the symmetric linear (harmonic) system. Interestingly, even when two couplings are asymmetric,  $\gamma_2 \neq \gamma_1$ , we find  $\partial_{i\chi} \mathcal{F}_{T_2 T_1}|_{\chi=0} = 0$  in spite of nonzero  $\mathcal{F}_{T_2 T_1}$ , so that the geometric heat flux is still vanishing,  $J_{\text{geom}} \equiv \partial_{i\chi} G_{\text{geom}}|_{\chi=0} = 0$ . In other words, although the geometric effect exists in the classical asymmetric linear system, its contribution is not observable if one measures only the average flux. The geometric effect can only manifest itself in the higher order heat fluctuations, like the shot noise of currents, i.e.,  $\partial_{i\chi}^2 G_{\text{geom}}|_{\chi=0} \propto \partial_{i\chi}^2 \mathcal{F}_{T_2 T_1}|_{\chi=0} \neq 0$ .

Recalling the modulation  $(\gamma_2, T_1)$  can induce the nonzero geometric heat flux, we speculate that as long as the viscosity  $\gamma_2$  is temperature dependent, the modulation  $(T_2, T_1)$  can be effectively equal to the modulation  $(\gamma_2, T_1)$ . As a consequence, in such systems of temperature-dependent viscosity, merely modulating  $(T_2, T_1)$  may produce nonzero curvature thus nonzero geometric heat flux. In fact, viscosity is generally temperature dependent [31]. Phenomenologically, we can assume  $\gamma_j = \gamma_0 + aT_j^n$ , with  $j = 1, 2$ . Thus for the typical modulation  $(T_2, T_1)$ , we indeed obtain the nonzero geometric heat flux

$$J_{\text{geom}} = \frac{1}{T_p} \iint_{T_2 T_1} dT_2 dT_1 \frac{an(\gamma_0 + aT_1^n)(\gamma_0 + aT_2^n)(T_1^{n-1} + T_2^{n-1})}{2(2\gamma_0 + aT_1^n + aT_2^n)^3}. \quad (15)$$

Clearly, the temperature dependence plays a key role for the manifestation of geometric effect. Given a modulation cycle, the geometric heat flux  $J_{\text{geom}}$  vanishes when the temperature dependency  $a \rightarrow 0$ , while  $J_{\text{geom}}$  increases to a saturated value as  $a$  increases. When  $n = 0$ ,  $\gamma_j$  also becomes temperature independent, and in turn  $J_{\text{geom}}$  becomes zero as well.

Considering the intrinsic effective temperature dependencies of system parameters are ubiquitous in nonlinear interacting oscillators [32,33], we thus speculate that with the help of nonlinearity, the existing geometric

effect is able to manifest itself into the geometric heat flux as well. Therefore, we further consider the FPU- $\beta$  model [10], with nonlinear interacting potential:  $V(x_1, x_2) = \frac{k_1}{2}(x_1 - x_2)^2 + \frac{k_2}{4}(x_1 - x_2)^4$ . As we shall see in what follows, the nonlinear strength  $k_2$  can induce effective temperature dependence of system parameters [32,33] and is the crucial ingredient to manifest geometric heat flux.

In Fig. 2, we numerically simulate Eq. (1) and calculate the average geometric heat  $Q_p \equiv J_{\text{geom}} T_p$ , defined for a

driving cycle. Given the temperature driving protocol  $T_2 = 0.09 + 0.06 \cos(2\pi t/T_p + \pi/4)$ ,  $T_1 = 0.09 + 0.06 \sin(2\pi t/T_p + \pi/4)$ , we have a zero dynamic flux zero but a nonzero  $Q_p$ . When  $T_p$  becomes large (adiabatic limit),  $Q_p$  saturates to a fixed value independent of  $T_p$ , which indicates that it is purely a geometric property. The geometric heat per driving cycle does not rely on the driving rate, but only depends on the geometry of the driving contour in parameter spaces. The deviations of  $Q_p$  from the fixed value at the fast driving regime are due to the breakdown of the adiabatic precondition.

The inset of Fig. 2 verifies that when the nonlinear interaction reduces to harmonic coupling ( $k_2 = 0$ ), the geometric contribution disappears ( $J_{\text{geom}} = 0$ ). Only with the help of nonlinearity, the geometric effect of temperature modulations can manifest itself into the heat flux. Moreover, increasing  $k_2$  can enhance  $Q_p$ , until to a saturated value, which coincides qualitatively with the behavior of the analytic result Eq. (15), by increasing  $a$ . In fact, from the viewpoint of nonequilibrium Green's functions [34], the nonlinear interaction effect in thermal transport is reflected in the temperature-dependent effective self-energies, which in our case are exactly the temperature dependent  $\gamma_i$ .

Although we focus on a two-coupled-oscillator system at the moment, it could be straightforward to generalize the above analysis into arbitrarily long coupled-oscillator model with inertial terms, of which the eigenvalues and eigenvectors of the twisted Fokker-Planck operator can be obtained in terms of appropriate phonon Green's functions [25].

Now, we shall examine the previous studies to see whether it is justified for neglecting geometric heat flux. In [3,4], only one bath temperature is under cyclic manipulation so that the absence of geometric heat flux is justified, because for cyclic driving, the manifestation of geometric effect requires at least two parameter modulations in order to enclose a nonzero area in the parameter space. The same is true in [7], where the interacting lattice is only cyclically driven by one mechanical force. In [4–6], although two

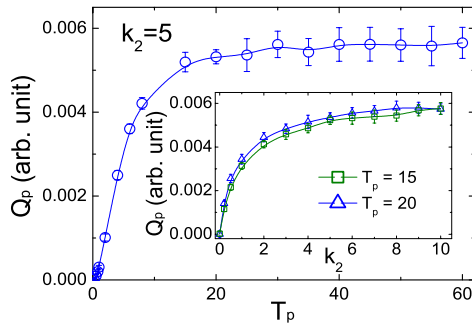


FIG. 2 (color online). Nonlinear effect on manifestation of geometric effect in classical heat transport for FPU- $\beta$  model.  $k_1 = 0.5$ ,  $\gamma_1 = \gamma_2 = 5$ . The physical value correspondences of those dimensionless units can be found in Ref. [10]. The error bar denotes the standard derivation of 25 times simulations, each is averaged over  $10^6$  periods.

bath temperatures are modulated, they are varying either isothermally  $T_1 = T_2 = T_0 + \Delta T(t)$  [5] or inversely  $T_1 = T_0 + \Delta T(t)$ ,  $T_2 = T_0 - \Delta T(t)$  [4,6]. In this way, the modulation contour only closes as a line segment. Therefore, the absence of geometric heat flux is also justified.

In the last part, we would like to propose an experimental implementation to demonstrate our predictions on manifestation of geometric effect in classical heat transport through coupled oscillators. In view of the well-known electric analogy of interacting oscillators' Brownian motion [23,35], we are able to map the oscillator system into a  $RC$  circuit, where two resistors of resistance  $R_j$  are arranged in parallel with a capacitor of capacitance  $C$ , as shown in Fig. 3(a). The left (right) circuit part is subjected to a thermal reservoirs of temperature  $T_1$  ( $T_2$ ), which generates a Gaussian voltage fluctuation  $\delta V_1$  ( $\delta V_2$ ), so called Johnson-Nyquist noise, with variance  $\langle \delta V_i(t) \delta V_j(t') \rangle = 2R_i T_i \delta_{ij} \delta(t - t')$  [23,35].  $q_j$  denotes the charge going through the resistor  $R_j$  and  $dq_j/dt$  is the corresponding electric current. Consequently, the dynamics of this  $RC$  circuit is described by

$$\begin{aligned} R_1 dq_1/dt + (q_1 - q_2)/C &= \delta V_1, \\ R_2 dq_2/dt + (q_2 - q_1)/C &= \delta V_2. \end{aligned} \quad (16)$$

Note that Eq. (16) is of a similar form as the overdamped dynamics Eq. (1) for interacting oscillators. The heat trans-

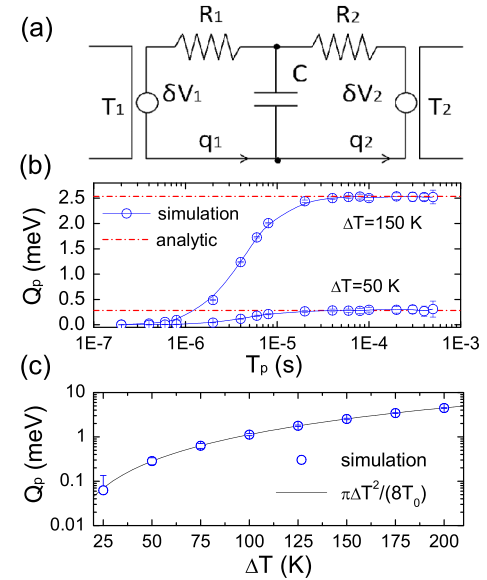


FIG. 3 (color online). (a) A parallel  $RC$  electric circuit subject to Johnson-Nyquist noises. The analogy with two coupled Brownian oscillators with heat baths is reflected by the parameter correspondences:  $(q_j, R_j, 1/C, \delta V_j) \leftrightarrow (x_j, \gamma_j, k, \xi_j)$ . (b) Manifestation of geometric effect for classical heat transport in  $RC$  electric circuit with temperature-dependent resistances. (c)  $Q_p = J_{\text{geom}} T_p$  as a function of  $\Delta T$ . The simulations are well fitted by  $\pi \Delta T^2 / (8T_0)$ , approximated from Eq. (17). Parameters are  $a = 10 \text{ k}\Omega/\text{K}$ ,  $C = 1 \text{ pF}$ ,  $T_0 = 300 \text{ K}$ .

ferred over time  $t$  now is analogously defined as  $Q(t) = \int_0^t \dot{q}_1(q_1 - q_2)/Cd t'$ , which actually is the charging work done by the left reservoir on the capacitor. Thereafter, we can modulate  $C(t)$ ,  $R_j(t)$ ,  $T_j(t)$  to test our predictions.

Nonlinear capacitor can be used to simulate the nonlinearity effect. To mimic the related temperature-dependent viscosity, we can choose resistors of temperature-dependent resistance, e.g.,  $R_j = aT_j$ , with contacting them to their respective reservoirs. From Eq. (15), we then have

$$J_{\text{geom}} = \frac{1}{T_p} \iint_{T_2 T_1} dT_2 dT_1 \frac{T_1 T_2}{(T_1 + T_2)^3}. \quad (17)$$

For cyclic modulation:  $T_2 = T_0 + \Delta T \cos(2\pi t/T_p + \pi/4)$ ,  $T_1 = T_0 + \Delta T \sin(2\pi t/T_p + \pi/4)$ , we will have zero  $J_{\text{dyn}}$  but nonzero  $J_{\text{geom}}$ . Although this geometric heat flux is independent of  $C$  and  $R_j$ , the adiabatic precondition Eq. (5) requires  $T_p \gg T_c = CR_1 R_2 / (R_1 + R_2)$ . Assume we use  $a = 10 \text{ k}\Omega/\text{K}$ ,  $C = 1 \text{ pF}$ ,  $T_0 = 300 \text{ K}$ , we will have the adiabatic condition  $T_p \gg T_c \approx 3 \mu\text{s}$ . Analytic results are verified by numerical simulations of Eq. (16), as plotted in Fig. 3(b). Clearly, for  $T_p = 20 \mu\text{s}$ , the geometric heat per cycle already reaches the adiabatic limit. Therefore, as long as the system evolves a long time, i.e., with large number of cycles, we can accumulate a large geometric heat, e.g., for  $T_p = 20 \mu\text{s}$ ,  $\Delta T = 50 \text{ K}$ ,  $Q_p = 0.283 \text{ meV}$ , after each minute we can have 847 ev. Moreover, if we scale up many  $RC$  circuits in parallel, we are able to obtain even larger geometric heat flux. Alternatively, by increasing the modulation amplitude, one can also increase geometric heat flux, as shown in Fig. 3(c).

Given the fact that the fluctuation theorems in electric circuits [36] as well as the energy rectification have been experimentally realized in a nonlinear electrical transmission line [37], we believe that our prediction of geometric-phase effect on heat transport can be also experimentally validated in a foreseeable future. Furthermore, as our ability to design and manipulate nano or micro-sized systems improves, we believe that the present study of geometric energy (heat) flux could provide a new means of energy harvesting by harnessing the ubiquitous cyclic changes in the Universe.

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- [1] D. Segal and A. Nitzan, *Phys. Rev. E* **73**, 026109 (2006); D. Segal, *Phys. Rev. Lett.* **101**, 260601 (2008).
- [2] R. Marathe, A. M. Jayannavar, and A. Dhar, *Phys. Rev. E* **75**, 030103(R) (2007).
- [3] N. Li, P. Hänggi, and B. Li, *Europhys. Lett.* **84**, 40009 (2008).
- [4] N. Li, F. Zhan, P. Hänggi, and B. Li, *Phys. Rev. E* **80**, 011125 (2009).
- [5] J. Ren and B. Li, *Phys. Rev. E* **81**, 021111 (2010).
- [6] A. Dhar, O. Narayan, A. Kundu, and K. Saito, *Phys. Rev. E* **83**, 011101 (2011).
- [7] S. Zhang, J. Ren, and B. Li, *Phys. Rev. E* **84**, 031122 (2011); B. Q. Ai, D. He, and B. Hu, *Phys. Rev. E* **81**, 031124 (2010).
- [8] E. C. Cuansing and J.-S. Wang, *Phys. Rev. E* **82**, 021116 (2010); B. K. Agarwalla, J.-S. Wang, and B. Li, *Phys. Rev. E* **84**, 041115 (2011).
- [9] J. Ren, P. Hänggi, and B. Li, *Phys. Rev. Lett.* **104**, 170601 (2010).
- [10] N. Li *et al.*, [arXiv:1108.6120](https://arxiv.org/abs/1108.6120) [*Rev. Mod. Phys.* (to be published)].
- [11] D. Xiao, M.-C. Chang, and Q. Niu, *Rev. Mod. Phys.* **82**, 1959 (2010).
- [12] E. Prodan and C. Prodan, *Phys. Rev. Lett.* **103**, 248101 (2009).
- [13] J.-T. Lü, M. Brandbyge, and P. Hedegård, *Nano Lett.* **10**, 1657 (2010).
- [14] L. Zhang, J. Ren, J.-S. Wang, and B. Li, *Phys. Rev. Lett.* **105**, 225901 (2010).
- [15] J. M. R. Parrondo, *Phys. Rev. E* **57**, 7297 (1998).
- [16] Y. Shi and Q. Niu, *Europhys. Lett.* **59**, 324 (2002).
- [17] R. D. Astumian and P. Hänggi, *Phys. Today* **55**, 33 (2002); R. D. Astumian, *Proc. Natl. Acad. Sci. U.S.A.* **104**, 19715 (2007); S. Rahav, J. Horowitz, and C. Jarzynski, *Phys. Rev. Lett.* **101**, 140602 (2008).
- [18] N. A. Sinitsyn and I. Nemenman, *Europhys. Lett.* **77**, 58001 (2007); *Phys. Rev. Lett.* **99**, 220408 (2007).
- [19] N. A. Sinitsyn, *J. Phys. A* **42**, 193001 (2009).
- [20] A. Dhar, *Adv. Phys.* **57**, 457 (2008); S. Lepri, R. Livi, and A. Politi, *Phys. Rep.* **377**, 1 (2003).
- [21] J. M. R. Parrondo and P. Español, *Am. J. Phys.* **64**, 1125 (1996).
- [22] E. M. Purcell, *Am. J. Phys.* **45**, 3 (1977).
- [23] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, North-Holland, 2007).
- [24] P. Visco, A. Puglisi, A. Barrat, E. Trizac, and F. van Wijland, *J. Stat. Phys.* **125**, 533 (2006).
- [25] A. Kundu, S. Sabhapandit, and A. Dhar, *J. Stat. Mech.* (2011) P03007.
- [26] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.108.210603> for details.
- [27] M. V. Berry, *Proc. R. Soc. A* **392**, 45 (1984).
- [28] J. G. Garrison and E. M. Wright, *Phys. Lett. A* **128**, 177 (1988).
- [29] V. Y. Chernyak, M. Chertkov, and N. A. Sinitsyn, *J. Stat. Mech.* (2011) P09006.
- [30] J. M. Horowitz and C. Jarzynski, *J. Stat. Phys.* **136**, 917 (2009); Jie Ren, V. Y. Chernyak, and N. A. Sinitsyn, *J. Stat. Mech.* (2011) P05011.
- [31] J. C. Maxwell, *Maxwell on Molecules and Gases*, edited by E. Garber, S. G. Brush and C. W. F. Everitt (The MIT Press, Cambridge, MA, 1986).
- [32] N. Li and B. Li, *Europhys. Lett.* **78**, 34001 (2007).
- [33] D. He, S. Buyukdagli, and B. Hu, *Phys. Rev. E* **78**, 061103 (2008).
- [34] J.-S. Wang, J. Wang, and J. T. Lü, *Eur. Phys. J. B* **62**, 381 (2008).
- [35] R. van Zon, S. Ciliberto, and E. G. D. Cohen, *Phys. Rev. Lett.* **92**, 130601 (2004).
- [36] N. Garnier and S. Ciliberto, *Phys. Rev. E* **71**, 060101(R) (2005).
- [37] F. Tao, W. Chen, W. Xu, J. Pan, and S. Du, *Phys. Rev. E* **83**, 056605 (2011).