# Nonlocality of Symmetric States

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In this Letter we study the nonlocal properties of permutation symmetric states of $n$ qubits. We show that all these states are nonlocal, via an extended version of the Hardy paradox and associated inequalities. Natural extensions of both the paradoxes and the inequalities are developed which relate different entanglement classes to different nonlocal features. Belonging to a given entanglement class will guarantee the violation of associated Bell inequalities which see the persistence of correlations to subsets of players, whereas there are states outside that class which do not violate.

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*Introduction.*—Nonlocality is a foundational feature of quantum mechanics and is increasingly becoming recognized as a key resource for quantum information theory, for example, in device independence [1–4], communication complexity [5], and measurement based quantum computation [6,7]. Related, though not equivalent is the feature of entanglement. In the multipartite setting entanglement is very complicated, different classes of entanglement exist, each having potentially different roles as resources. Very little is currently known if, and how, the richness of multipartite entanglement is reflected in nonlocal features, with some very recent breakthroughs [8].

We explore the nonlocal features of permutation symmetric states of qubits. This set of states are useful in a variety of quantum information tasks, they occur naturally as ground states in some Bose-Hubbard models, and are among the most developed experimentally. Relatively little is known about the nonlocality of permutation symmetric states, mostly restricted to $W$ and GHZ states [9–11].

Knowing more about their nonlocality would help understand more their potential as resources for quantum information processing, and understand better the relationship between the subtleties of multipartite entangled states and nonlocal features. Recently we begin to get a better understanding of the entanglement features of symmetric states using the Majorana representation [12–17]. Here we use the same tool to study the states' nonlocality, allowing us to compare it to entanglement easily.

Consider $n$ parties, indexed by $i$, each of which make a measurement in a chosen setting $M_i$, and get result $r_i$. We will consider a choice of two settings, each with two outcomes. A probability distribution over the measurements is local or admits a local hidden variable (LHV) description if the joint probability distribution can be written as the product of individual probabilities given the value of some hidden variable $\lambda$:

$$P(r_1, \ldots, r_n|M_1, \ldots, M_n) = \int \rho(\lambda)\prod_i P(r_i|M_i, \lambda)d\lambda, \tag{1}$$

where $P(r_i|M_i, \lambda)$ is the probability for the $i$th party to obtain the result $r_i$ when using the measurement setting $M_i$ while having $\lambda$ as the value of the hidden variable. $\rho(\lambda)$ is the probability distribution of $\lambda$. $P(r_1, \ldots, r_n|M_1, \ldots, M_n)$ is the joint probability distribution when all $n$ parties measure using the settings $M_1, \ldots, M_n$ and obtain the results $r_1, \ldots, r_n$. Often we will ignore the lower index when position is obvious. It is obvious that local measurement on any separable state admits an LHV description. However, nonlocality does not follow directly from entanglement [18].

The Hardy paradox has been proposed as an ''almost probability-free'' test of the nonlocality of almost all bipartite entangled quantum states [19,20]. We first show that all permutation symmetric states of $n$ qubits can violate an extended version of the Hardy paradox and associated inequalities. While there exists earlier work generalizing the Hardy paradox to an $n$ party [21] equivalent to ours, we give a constructive way of finding measurements needed by the $n$-party paradox to show that all permutation symmetric states are nonlocal.

*The Hardy paradox and inequality for all permutation symmetric states.*—The original Hardy paradox consists of four probabilistic conditions that we impose on the outcomes of an experiment involving two parties [19,20]. These conditions are individually compatible with the definition of a hidden variable theory given in (1). But when taken together, they lead to a logical contradiction. Hardy showed that for almost all bipartite entangled states, there exist measurement settings to satisfy all these conditions, thus showing the incompatibility of LHV theory and quantum mechanics. The only exception is the maximally entangled states. Fortunately, the nonlocality of the maximally entangled states was proven before [22,23].

A multiparty extension of the Hardy paradox can be constructed as follows. First suppose there are $n$ players involved in an experiment. Each player can choose to measure one of two possible measurement settings labeled 0 or 1, and get one of two possible outcomes, also labeled 0 or 1. The first probabilistic condition we impose is that if everyone measures in the 0 basis, then sometimes everyone gets the result 0:

$$P(00\ldots00|00\ldots00) > 0. \qquad (2)$$

The next $n$ conditions are the same as above for $n-1$ players, but now if one player measures in the setting 1 instead of the setting 0 they will never get the result 0.

$$P(00\ldots00|\pi(00\ldots01)) = 0, \qquad (3)$$

for all permutations $\pi$ of bit strings with one 1 and $n-1$ zeros. Let us consider the implications imposed if these arise from an LHV model. We see that (1) and (2) imply that there exists at least one value of $\lambda$ such that $\forall\ i, P(0_i|0_i\lambda) > 0$. Then, for this particular value of $\lambda$, we know from (3) that $\forall\ i, P(0_i|1_i\lambda) = 0$. Since there are only two possible outcomes for each measurement setting, (3) imply that for this value of $\lambda$, should everyone instead chose to measure in setting 1, they must all get result 1 with certainty.

The last condition we impose contradicts the conclusion we get above. If everyone measures in setting 1, then they will never all get the result 1:

$$P(11\ldots1|11\ldots11) = 0. \qquad (4)$$

Clearly (2)–(4) are not possible within LHV. Note that in the case where $n = 2$, we recover the original Hardy paradox [19,20].

However, we will now give a constructive procedure to find the bases 0 and 1 for almost all permutation symmetric states such that the conditions (2) to (4) are all satisfied. As a prerequisite, let us recall some basic properties of permutation symmetric states and the Majorana representation. More details can be found in the Supplemental Material [24] and in [12,16,17,25,26].

A permutation symmetric state of $n$ qubits can be written in the form $|\psi\rangle = \sum_{k=0}^{n} c_k|S(n,k)\rangle$, where $|S(n,k)\rangle = \binom{n}{k}^{-1/2}\sum_{\text{perm}}|\underbrace{0\ldots0}_{n-k}\underbrace{1\ldots1}_{k}\rangle$ are Dicke states.

In the Majorana representation, the state $|\psi\rangle$ is written as a sum of permutations of the tensor product of $n$ qubits $\{|\eta_1\rangle\ldots|\eta_n\rangle\}$, called the Majorana points (MPs) of the state $|\psi\rangle$:

$$|\psi\rangle = K\sum_{\text{perm}}|\eta_1\ldots\eta_n\rangle. \qquad (5)$$

$K$ is a normalization constant which depends on the overlap between different MPs. In the Majorana representation, local unitaries of the form $U^{\otimes n}$ simply rotate all Majorana points at the same time, thus equivalent to a rotation of the Bloch sphere.

Permutation symmetry also persists to subspaces. If $|\psi\rangle$ is a permutation symmetric state of $n$ qubits, then for any single qubit state $|\chi\rangle$ (ignoring normalization), the $(n-1)$-qubit state $\langle\chi|\psi\rangle$ is also permutation symmetric:

$$\langle\chi|\psi\rangle = \sum_{i=1}^{n} C_i \sum_{\text{perm}}|\underbrace{\eta_1\ldots\eta_n}_{\{1,\ldots,n\}\setminus i}\rangle, \qquad (6)$$

where $C_i = \langle\chi|\eta_i\rangle$ and $\{1,\ldots,n\}\setminus i$ means that we discard the MP $|\eta_i\rangle$.

The equation below holds if and only if $|\eta_i\rangle$ is an MP of $|\psi\rangle$, and $|\eta_i^\perp\rangle$ is its antipodal point on the Bloch sphere:

$$(\langle\eta_i^\perp|)^{\otimes n}|\psi\rangle = 0. \qquad (7)$$

We will now see how to choose the measurement bases that satisfy (2)–(4) for almost all permutation symmetric states. First of all, (7) can be seen as the probability amplitude that gives (4) if we restrict the measurement to be projective and take the $\{|\eta_i\rangle, |\eta_i^\perp\rangle\}$ basis as measurement setting 1 for all parties.

For condition (3), if one can be satisfied, then by symmetry of the state the rest are satisfied automatically. Consider the projection of the state $|\psi\rangle$ on one of its MPs $|\eta_i\rangle$. By (6) this gives us a new permutation symmetric state of $(n-1)$ qubits:

$$|\psi'\rangle = \langle\eta_i|\psi\rangle = \sum_{j=1}^{n} C_j \sum_{\text{perm}}|\underbrace{\eta_1\ldots\eta_n}_{\{1,\ldots,n\}\setminus j}\rangle, \qquad (8)$$

with $C_j = \langle\eta_i|\eta_j\rangle$. The state $|\psi'\rangle$ has $(n-1)$ MPs, possibly different from the MPs of $|\psi\rangle$. In fact, the proposition below shows that for all permutation symmetric states except Dicke states, there is always at least one MP of $|\psi'\rangle$ that is different from all the MPs of $|\psi\rangle$.

*Proposition 1.*—Let $S_\psi := \{|\eta_1\rangle, |\eta_2\rangle, \ldots, |\eta_n\rangle\}$ be the set of MPs of the state $|\psi\rangle$. Let $S_{\psi_i} := \{|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_{n-1}\rangle\}$ be the set of MPs of the state $|\psi_i\rangle = \langle\eta_i|\psi\rangle$. Then $S_{\psi_i} \subseteq S_\psi iff |\psi\rangle$ is a Dicke state up to rotations of the Bloch sphere.

See Supplemental Material [24] for proof.

Let $|\mu_i\rangle$ be an MP of the state $|\psi'\rangle$ as defined in (8) that is different from all the MPs of $|\psi\rangle$, then by (7)

$$(\langle\mu_i^\perp|)^{\otimes n-1}|\psi'\rangle = \langle\eta_i\underbrace{\mu_i^\perp\ldots\mu_i^\perp}_{n-1}|\psi\rangle = 0. \qquad (9)$$

By choosing basis $\{|\mu_i\rangle, |\mu_i^\perp\rangle\}$ as measurement setting 0, for all parties, the probability amplitude (9) implies the satisfaction of condition (3) by symmetry. Because $|\mu_i\rangle$ is not an MP of $|\psi\rangle$, $\langle\underbrace{\mu_i^\perp\ldots\mu_i^\perp}_{n}|\psi\rangle \neq 0$. Thus (2) is satisfied automatically also. By Proposition 1, this procedure of choosing measurement settings 0 and 1 works for all permutation symmetric states except Dicke states.

The paradox itself, however, cannot be tested directly by experiments because in real experiments, when taking real-world noise and inaccuracy into account, we will never see probabilities getting exactly zero. To make the paradox more noise tolerant, we make it into an inequality. The LHV upper bound of the inequality can be violated by the amount of (2) when we use the procedure given above to perform a quantum experiment.

*Proposition 2.*—The Bell operator for $n$ systems

$$\mathcal{P}^n := P(0\ldots0|00\ldots00) - \sum_{\pi} P(00\ldots00|\pi(00\ldots01)) - P(1\ldots1|11\ldots11)$$

is bounded under LHV as $\mathcal{P}^n \leq 0$.

See Supplemental Material [24] for proof. (See also [21] for an alternative proof.)

Although the procedure we used to find measurement settings does not work for Dicke states, the inequality above can be violated by Dicke states in a 2 settings–2 outcomes experiment.

*Proposition 3.*—There exists an angle $0 < \theta < \pi$ such that all Dicke states $|S(n,k)\rangle$ $(\{k,n\} \in \mathbb{N}, 1 < k < n)$ violate the inequality in Proposition 2 when using the measurement setting $\{|+\rangle, |-\rangle\}$ as setting 0 and $\{\cos\frac{\theta}{2}|0\rangle - \sin\frac{\theta}{2}|1\rangle, \sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle\}$ as setting 1.

See Supplemental Material [24] for proof.

*Entanglement classes and nonlocality.*—In the Majorana representation of a permutation symmetric state (5), the Majorana points $\{|\eta_1\rangle, \ldots, |\eta_n\rangle\}$ are not necessarily all distinct. In this section we will slightly alter our notation to incorporate the notion of multiplicity or degeneracy, which means several MPs are "sitting on top of one another." In the new notation, we use $d_i$ to denote the degeneracy of the MP $|\eta_i\rangle$. So (5) becomes

$$|\psi\rangle = K \sum_{\text{perm}} |\eta_1^{d_1} \eta_2^{d_2} \ldots \eta_l^{d_l}\rangle, \qquad \forall\ i \neq j,$$
$$|\eta_i\rangle \neq |\eta_j\rangle, \qquad \sum_{i=1}^{l} d_i = n. \tag{10}$$

Also, (7) becomes

$$(\langle \eta_i^{\perp}|)^{\otimes k} |\psi\rangle = 0, \tag{11}$$

where $(n - d_i) < k \leq n$. Degeneracy of points cannot change under local operations and classical communication, even stochastically (SLOCC) [13] (see also [14,15]). Thus different degeneracies of points, corresponds to different entanglement classes.

Taking degeneracy into account, we can extend the paradox by considering subsets of players. Translating (11) to statements of probabilities, we see that the correlations of (4) persist to fewer players

$$P(\underbrace{11\ldots1}_{k}|\underbrace{11\ldots1}_{k}) = 0, \tag{12}$$

for $(n - d_i) < k \leq n$. The inequality in Proposition 2 can be extended naturally to

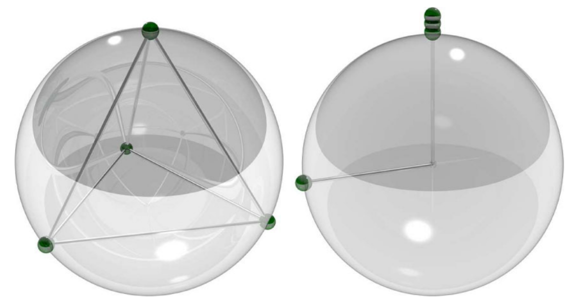
$$\mathcal{Q}_d^n := \mathcal{P}^n - P(\underbrace{11\ldots1}_{n-1}|\underbrace{11\ldots1}_{n-1}) - \ldots - P(\underbrace{11\ldots1}_{n-d+1}|\underbrace{11\ldots1}_{n-d+1}) \leq 0. \tag{13}$$

The LHV upper bound holds because $\mathcal{P}^n$ is negative by Proposition 2, and we are only subtracting positive probabilities from it.

We can now see how this inequality allows us to differentiate different entanglement classes via degeneracy classes. First, it is clear that all states with at least one MP with degeneracy $d$ will be able to violate inequality (13). Second, this is not true for all states with lower degeneracy. Note that we cannot hope that all states with maximum degeneracy less than $d$ do not violate $Q_d^n$, or indeed any inequality violated by all states with degeneracy $d$, since we can lower the degeneracy by moving one MP away by an arbitrary distance. In this sense the best that we could hope for is that certain states, or classes of states outside the associated entanglement class cannot violate. It can be checked by using semidefinite programming techniques similar to the ones used in [8] that the tetrahedron state, shown in Fig. 1(a) does not violate $Q_3^4$ while the state in Fig. 1(b) does. A similar separation between $W$ states and Schmidt states (states such that removal of one system destroys the entanglement) has been found recently in [8].

Furthermore, if all parties measure projectively in the same basis, the only way they can satisfy conditions (3)–(5) as well as (12) is if they have at least one MP with degeneracy $d \geq d_i$. This is because (5) implies the basis is an MP, and (12) bounds the degeneracy of it. In this sense one can witness different entanglement classes. One may further expect that with the same measurement restrictions high violation of the inequalities depends on the degeneracies. In this way these extensions probe the different entanglement types given be differing degenerecies [13,27].

*Conclusions.*—In this Letter we have presented new Hardy type paradoxes and associated Bell inequalities, and given a procedure to find bases to show violation for all permutation symmetric states of qubits, which can be understood as the generalization of Gisin's theorem [28] to permutation symmetric states. One property of the inequalities which is obvious immediately is that the inequalities are written in terms of probabilities and cannot be extended to normal correlation operators alone. Since the number of



(a) $Q_3^4 \leq -0.0609$ (b) $Q_3^4 \geq 0.0141$

FIG. 1 (color online). The state (a) $|T\rangle = \sqrt{1/3}|0000\rangle + \sqrt{2/3}|S(4,3)\rangle$ does not violate $Q_3^4$ while all states with degeneracy $d = 3$ do, such as the state (b) $|D_3\rangle = K\sum_{\text{perm}}|000+\rangle$.

settings and outcomes is two, and since it also works for all extended GHZ states, it provides an example of a Bell operator which is more powerful than possible by correlation operators alone [29].

The structure of nonlocal features is also explored via these methods. Natural extensions of both the paradox and the inequalities are presented which relate to different entanglement classes (specifically, degeneracy classes [13]). On the one hand this provides a witness to discriminate different entanglement classes if measurements are set as outlined. On the other hand, states of minimum degeneracy can certainly violate associated inequalities, whereas other states will not, for example, the state $|T\rangle$ as shown here, no matter what measurements are made, hence providing a possibility for device independent testing of state class, as was done in [8] for $W$ and Schmidt state classes. Thus high degeneracy of MPs, originally considered in terms of the abstract definition of SLOCC classification, has a practical application demonstrating the persistence of nonlocal correlations to fewer systems. In this sense, the difference noted between $W$ and GHZ states in [10] are just examples of this more general feature.

As nonlocal features become more and more recognized as important for applications in quantum information we can expect that these results will lead to better understanding of the usefulness of permutation symmetric states. We also note that preparation of these states, and indeed the projective measurements presented is well within experimental grasp in several different possible experimental frameworks [30–32].

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