

Probing the Equation of State of Nuclear Matter via Neutron Star Asteroseismology

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We general-relativistically calculate the frequency of fundamental torsional oscillations of neutron star crusts, where we focus on the crystalline properties obtained from macroscopic nuclear models in a way that is dependent on the equation of state of nuclear matter. We find that the calculated frequency is sensitive to the density dependence of the symmetry energy, but almost independent of the incompressibility of symmetric nuclear matter. By identifying the lowest-frequency quasiperiodic oscillation in giant flares observed from soft gamma-ray repeaters as the fundamental torsional mode and allowing for the dependence of the calculated frequency on stellar models, we provide a lower limit of the density derivative of the symmetry energy as $L \simeq 50$ MeV.

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Observations of the global oscillations of stars play a significant role in probing the properties of stellar matter, as in the case of the Sun. Studies in this direction are often referred to as asteroseismology. Oscillations of neutron stars are expected to give an insight to the properties of matter under conditions of ultrahigh density. Recent observations of the quasiperiodic oscillations (QPOs) in giant flares from soft gamma-ray repeaters (SGRs) [1] suggest that long-awaited evidence for the neutron star oscillations has been given with the help of coupling of gigantic magnetic fields with the solid crust. So far, three giant flares have been detected from SGR 0526-66, SGR 1900 + 14, and SGR 1806-20, and, through the timing analysis of the x-ray afterglow, the QPOs with frequencies in the range from tens of hertz up to a few kilohertz have been discovered [1]. Many theoretical attempts to explain the observed QPO frequencies have been done in terms of the torsional oscillations in the crustal region and/or the magnetic oscillations (e.g., Refs. [2–9]). If the QPOs in giant flares are associated with the crustal oscillations, the properties of inhomogeneous nuclear matter in the crust could be clarified [10–12].

The outer part of neutron stars can be described as follows. Below an ionic ocean in the vicinity of the surface, a bcc Coulomb lattice of nuclei embedded in a roughly uniform electron gas is considered to compose a crustal region. In this region, nuclei become gradually neutron rich with increasing density and even drip neutrons at a density of about 4×10^{11} g cm⁻³. Near normal nuclear density, this crustal region is considered to melt into uniform nuclear matter. Just before melting, roughly spherical nuclei are so closely packed that global deformations into rodlike nuclei could occur. As the density increases further, possible changes of the nuclear shape are rod, slab, tube, and bubble [13,14]. These exotic nuclei are often called

“nuclear pasta.” Not only the density region of the pasta phases but also the charge number of roughly spherical nuclei is known to be sensitive to the empirically uncertain density dependence of the symmetry energy [15,16]. Generally, it is difficult to observationally probe the properties of matter in the crust, but asteroseismology could exceptionally help to constrain the matter properties, such as nuclei present and the equation of state (EOS) (e.g., Refs. [17–20]). In this Letter, we will show by systematic analyses that an approach to the QPOs in SGR giant flares in terms of the torsional shear modes in the crust could severely constrain the density dependence of the symmetry energy.

We begin with the bulk energy per nucleon near the saturation point of symmetric nuclear matter at zero temperature, which can be written as a function of nucleon density n and neutron excess α [21],

$$w = w_0 + \frac{K_0}{18n_0^2}(n - n_0)^2 + \left[S_0 + \frac{L}{3n_0}(n - n_0) \right] \alpha^2, \quad (1)$$

where w_0 , n_0 , and K_0 are the saturation energy, saturation density, and incompressibility of symmetric nuclear matter, respectively. The parameters L and S_0 characterize the symmetry energy coefficient $S(n)$: $S_0 = S(n_0)$ is the symmetry energy coefficient at $n = n_0$, while $L = 3n_0(dS/dn)_{n=n_0}$ is the symmetry energy density derivative coefficient. The parameters w_0 , n_0 , and S_0 can be relatively easier to determine from empirical masses and radii of stable nuclei [22]. On the other hand, the remaining two parameters, L and K_0 , are more difficult to determine. Here we introduce a new parameter y defined as $y = -K_0 S_0 / (3n_0 L)$, which denotes the slope of the saturation line in the vicinity of $\alpha = 0$ [22]. Two of us (K. O. and K. I.) constructed the model for the EOS of nuclear matter

in such a way as to reproduce Eq. (1) in the limit of $n \rightarrow n_0$ and $\alpha \rightarrow 0$, calculated the optimal density distribution of stable nuclei within a simplified version of the extended Thomas-Fermi theory, and obtained the values of w_0 , n_0 , and S_0 for given y (or L) and K_0 by fitting the charge number, mass excess, and charge radius that can be calculated from the optimal density distribution to the empirical behavior. In a manner similar to Ref. [15], we adopt the parameter range satisfying $0 < L < 160$ MeV, $180 \text{ MeV} \leq K_0 \leq 360$ MeV, and $y < -200 \text{ MeV fm}^3$. In fact, this parameter range reproduces equally well the mass and radius data for stable nuclei and effectively covers even extreme cases [22]. With such parameters, we can obtain the crust EOS and the equilibrium nuclear shape and size by generalizing the Thomas-Fermi model for nuclei to matter in the crust. The results for the nucleon densities n_1 at which the nuclear shape changes from spherical to cylindrical, and n_2 at which the nuclear matter becomes uniform, were tabulated for several sets of y (or L) and K_0 in Table I. Note that the interval between n_1 and n_2 , which corresponds to the pasta region, decreases with L and vanishes at $L \sim 100$ MeV.

We turn to the equilibrium structure of nonrotating neutron stars, which is determined by the Tolman-Oppenheimer-Volkoff (TOV) equations for given EOS of neutron star matter. For a solution to these equations, the metric can be described in terms of the spherical polar coordinates r , θ , and ϕ as

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2)$$

However, the EOS for the core surrounded by the crust is still uncertain, although there are many calculations based on realistic nuclear interactions and even constraints from the observationally deduced neutron star masses and radii [23,24]. To avoid this uncertainty, we here construct the crust with the EOS models mentioned above by solving the TOV equations inward from the surface of the star for given stellar mass M and radius R as in Ref. [25]. We remark that the thickness of the crust thus constructed is consistent

TABLE I. The pasta density region calculated for several sets of the EOS parameters.

y (MeV fm ³)	K_0 (MeV)	L (MeV)	n_1 (fm ⁻³)	n_2 (fm ⁻³)
-220	180	52.2	0.060	0.079
-220	230	73.4	0.064	0.073
-220	360	146.1	0.066	0.066
-350	180	31.0	0.058	0.091
-350	230	42.6	0.063	0.086
-350	360	76.4	0.072	0.076
-1800	180	5.7	0.058	0.134
-1800	230	7.6	0.058	0.127
-1800	360	12.8	0.058	0.118

with a typical behavior given in terms of M and R by Eq. (18) in Ref. [26].

We next consider the shear modulus of a crustal part composed of spherical nuclei of charge Ze and number density n_i , which can be approximately described as [27]

$$\mu = 0.1194n_i(Ze)^2/a, \quad (3)$$

where $a = (3/4\pi n_i)^{1/3}$ is the Wigner-Seitz radius. Note that this formula is derived in the limit of zero temperature from Monte Carlo calculations of the shear modulus averaged over all directions for a perfect bcc Coulomb crystal of point charges embedded in a neutralizing uniform background [28]. We will use this formula for calculations of the torsional oscillation frequencies since effects of quantum zero-point motions and thermal fluctuations are negligible. For pasta nuclei, except bubbles, the elasticity is expected to be much lower than that for spherical nuclei [29]. Moreover, the core, which is expected to be composed mostly of fluids, may have a structure with non-vanishing elasticity [30]. Here, we simply assume $\mu = 0$ at $n > n_1$, as in Ref. [12]. Within this assumption, the shear modulus at $n > n_1$ is underestimated, which tends to lower the torsional oscillation frequencies. The frequencies as will be estimated below should thus be regarded as lower limits basically, but still will be shown to play a role in constraining L .

Since the torsional oscillations on a spherically symmetric star are incompressible, the star is free from deformation and density variation during such oscillations. One can thus determine the frequencies of the torsional oscillations with satisfactory accuracy even if one neglects the resulting metric perturbations by setting $\delta g_{\mu\nu} = 0$, which is known as the relativistic Cowling approximation. Within this approximation, the torsional oscillations can be described by a single perturbation variable, i.e., the angular displacement of the stellar matter \mathcal{Y} which is related to the ϕ component of the perturbed 4-velocity of a matter element δu^ϕ by $\delta u^\phi = e^{-\Phi} \partial_t \mathcal{Y}(t, r) \partial_\theta P_\ell(\cos\theta) / \sin\theta$ with the ℓ th order Legendre polynomial P_ℓ . Assuming $\mathcal{Y}(t, r) = e^{i\omega t} \mathcal{Y}(r)$, one can obtain the perturbation equation for $\mathcal{Y}(r)$ from the linearized equation of motion as [31]

$$\mathcal{Y}'' + \left[\left(\frac{4}{r} + \Phi' - \Lambda' \right) + \frac{\mu'}{\mu} \right] \mathcal{Y}' + \left[\frac{\epsilon + p}{\mu} \omega^2 e^{-2\Phi} - \frac{(\ell + 2)(\ell - 1)}{r^2} \right] e^{2\Lambda} \mathcal{Y} = 0, \quad (4)$$

where ϵ and p are the energy density and pressure, and the prime denotes the derivative with respect to r . To determine the eigenfrequencies, we adopt the zero-traction condition at $n = n_1$ and the zero-torque condition at the star's surface. These boundary conditions reduce to $\mathcal{Y}' = 0$ at $n = 0, n_1$ [5,31]. We remark that neutron superfluidity ignored here would subtract the mass density of superfluid neutrons from the enthalpy density $\epsilon + p$ and hence enhance the eigenfrequencies [32].

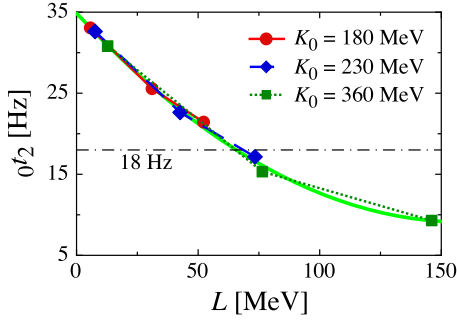


FIG. 1 (color online). Frequency of the $\ell = 2$ fundamental torsional oscillations, ${}_0t_2$, plotted as a function of L for $M = 1.4M_\odot$ and $R = 12$ km. The horizontal dot-dashed line denotes the lowest QPO frequency observed from SGR 1806-20 [1], while the solid thick line denotes the fitting formula (5).

First, to see the dependence of the fundamental torsional oscillations on the EOS parameters, we calculate the corresponding eigenfrequency for a typical stellar model with $M = 1.4M_\odot$ and $R = 12$ km. Figure 1 shows the frequency of the $\ell = 2$ fundamental torsional oscillations ${}_0t_2$ calculated for the nine sets of L and K_0 that are tabulated in Table I, together with the lowest QPO frequency in SGR 1806-20 [1]. From this figure, one can observe that ${}_0t_2$ is almost independent of K_0 in the parameter range adopted here once the stellar model is fixed. Such independence can be seen for various stellar models ranging $R = 10, 12,$ and 14 km as well as $M = 1.4M_\odot$ and $1.8M_\odot$. We can thus focus on the L dependence of the calculated ${}_0t_2$, which arises mainly because the nuclear charge Z decreases with L through the surface property [15], leading to decrease in the shear modulus (3) with L . Since Z depends strongly on L in the vicinity of $n = n_1$, this region is likely to play a role in constraining L via the evaluations of ${}_0t_2$. To see the L dependence in a continuous manner, we derive a fitting formula for ${}_0t_2$ by assuming the polynomial form,

$${}_0t_2 = c_0 - c_1L + c_2L^2, \quad (5)$$

where c_0 , c_1 , and c_2 are the adjustable parameters that depend on M and R . In fact, as can be seen in Fig. 1, this formula well reproduces the dependence of ${}_0t_2$ on L .

The fitting formula thus obtained is exhibited in Fig. 2. We find from this figure that ${}_0t_2$ for fixed L can be determined within the accuracy of $\sim 20\%$, if R is in the range of 10–14 km. By shifting M from $1.4M_\odot$ up to $1.8M_\odot$, we find that ${}_0t_2$ for $R = 10, 12,$ and 14 km decreases only by $\sim 14, 10,$ and 9% , respectively. For clarity, we plot, in Fig. 3, ${}_0t_2$ estimated for stellar models ranging $10 \text{ km} \leq R \leq 14 \text{ km}$ and $1.4M_\odot \leq M \leq 1.8M_\odot$. The results are confined within the painted region, which is so narrow that we can constrain L as we shall see.

Let us assume that the observed QPOs in SGR giant flares arise from the torsional oscillations in neutron star crusts and note that among many eigenfrequencies of the

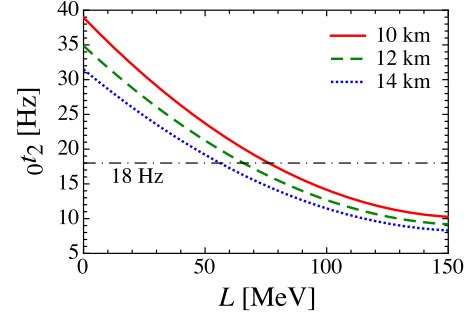


FIG. 2 (color online). ${}_0t_2$ given by formula (5) for $M = 1.4M_\odot$ and $R = 10, 12, 14$ km.

torsional oscillations ${}_0t_2$ is the lowest. Then, ${}_0t_2$ would become equal to or even lower than the lowest frequency in the observed QPOs. Consequently, one can constrain L as $L \geq 50$ MeV from the painted region in Fig. 3. Recall that the calculated ${}_0t_2$ is likely to be underestimated because of the simplified treatments of the shear modulus and the enthalpy density. In particular, elasticity in the pasta phases ignored here would act to increase ${}_0t_2$ as long as $n_2 > n_1$ [12]. At $L \geq 50$ MeV, however, $n_2 - n_1$ is already small (see Table I). The resultant modifications on the L constraint are thus expected to be small.

We proceed to show the frequency of the first overtone of the $\ell = 2$ torsional oscillations ${}_1t_2$ calculated again for the nine sets of L and K_0 that are tabulated in Table I. The results with $M = 1.4M_\odot$ and $R = 10, 12, 14$ km are plotted in Fig. 4. We find from this figure that ${}_1t_2$ depends not only on L , but on K_0 and R significantly, in contrast to the case of ${}_0t_2$. The R dependence arises because basically ${}_1t_2$ is inversely proportional to the crust thickness, which in turn increases as R^2 [26]. The K_0 dependence comes presumably from the K_0 dependence of n_1 (see Table I), but could be modified drastically once elasticity in the pasta phases is allowed for. This issue will be addressed elsewhere [33].

In summary, we have investigated the fundamental torsional mode and the first overtone in neutron star crusts for various EOS and stellar models. The identification of the lowest QPO frequency observed from SGR 1806-20 as ${}_0t_2$ would then allow us to constrain L as $L \geq 50$ MeV.

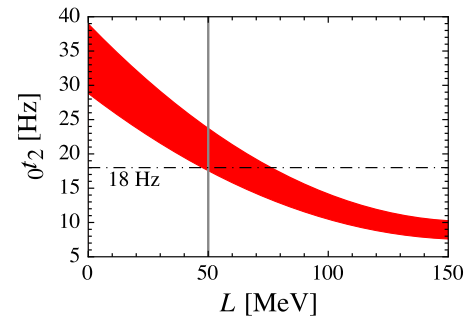


FIG. 3 (color online). Same as Fig. 2 for $10 \text{ km} \leq R \leq 14 \text{ km}$ and $1.4M_\odot \leq M \leq 1.8M_\odot$.

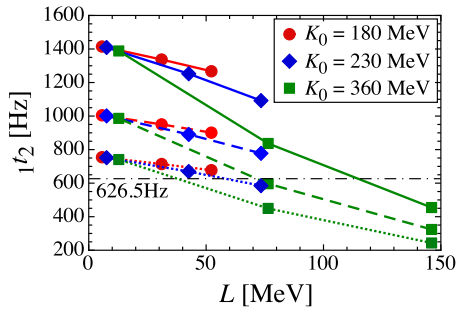


FIG. 4 (color online). Frequency of the first overtone of the $\ell = 2$ torsional oscillations, ${}_1t_2$, plotted as a function of L for $M = 1.4M_\odot$. The solid, broken, and dotted lines are the results for $R = 10, 12,$ and 14 km, while the circles, diamonds, and squares are those for $K_0 = 180, 230,$ and 360 MeV. The horizontal dot-dashed line denotes one of the high QPO frequencies observed from SGR 1806-20 [1].

At present, this constraint is fairly stringent because experimental constraints on L have yet to converge [34]. While there are earlier publications that remark the sensitivity of ${}_0t_2$ and ${}_1t_2$ to L [10,12], the present work is the first to provide a lower limit of L by sufficiently accurate and systematic calculations through the general-relativistic mode evaluations and the Thomas-Fermi treatment of nuclei. The present constraint already suggests that the pasta phases, if any in neutron stars, would occur in a narrow density region. Neutron superfluidity and pasta elasticity would make the constraint on L and the pasta region even more stringent.

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- [1] A.L. Watts and T.E. Strohmayer, *Adv. Space Res.* **40**, 1446 (2007).
- [2] Y. Levin, *Mon. Not. R. Astron. Soc.* **368**, L35 (2006).
- [3] U. Lee, *Mon. Not. R. Astron. Soc.* **374**, 1015 (2007).
- [4] L. Samuelsson and N. Andersson, *Mon. Not. R. Astron. Soc.* **374**, 256 (2007).
- [5] H. Sotani, K.D. Kokkotas, and N. Stergioulas, *Mon. Not. R. Astron. Soc.* **375**, 261 (2007); **385**, L5 (2008).
- [6] H. Sotani, A. Colaiuda, and K.D. Kokkotas, *Mon. Not. R. Astron. Soc.* **385**, 2161 (2008).

- [7] H. Sotani and K.D. Kokkotas, *Mon. Not. R. Astron. Soc.* **395**, 1163 (2009).
- [8] M. Gabler, P. Cerdá-Durán, J.A. Font, E. Müller, and N. Stergioulas, *Mon. Not. R. Astron. Soc.* **410**, L37 (2011).
- [9] A. Colaiuda and K.D. Kokkotas, *Mon. Not. R. Astron. Soc.* **414**, 3014 (2011).
- [10] A.W. Steiner and A.L. Watts, *Phys. Rev. Lett.* **103**, 181101 (2009).
- [11] H. Sotani, *Mon. Not. R. Astron. Soc.* **417**, L70 (2011).
- [12] M. Gearheart, W.G. Newton, J. Hooker, and B.A. Li, [arXiv:1106.4875](https://arxiv.org/abs/1106.4875) [*Mon. Not. R. Astron. Soc.* (to be published)].
- [13] C.P. Lorenz, D.G. Ravenhall, and C.J. Pethick, *Phys. Rev. Lett.* **70**, 379 (1993).
- [14] K. Oyamatsu, *Nucl. Phys. A* **561**, 431 (1993).
- [15] K. Oyamatsu and K. Iida, *Phys. Rev. C* **75**, 015801 (2007).
- [16] W.G. Newton, M. Gearheart, and B.A. Li, [arXiv:1110.4043](https://arxiv.org/abs/1110.4043).
- [17] N. Andersson and K.D. Kokkotas, *Phys. Rev. Lett.* **77**, 4134 (1996).
- [18] H. Sotani, K. Tominaga, and K. I. Maeda, *Phys. Rev. D* **65**, 024010 (2001).
- [19] H. Sotani, K. Kohri, and T. Harada, *Phys. Rev. D* **69**, 084008 (2004).
- [20] H. Sotani, N. Yasutake, T. Maruyama, and T. Tatsumi, *Phys. Rev. D* **83**, 024014 (2011).
- [21] J.M. Lattimer, *Annu. Rev. Nucl. Part. Sci.* **31**, 337 (1981).
- [22] K. Oyamatsu and K. Iida, *Prog. Theor. Phys.* **109**, 631 (2003).
- [23] F. Özel, G. Baym, and T. Güver, *Phys. Rev. D* **82**, 101301 (2010).
- [24] A. W. Steiner, J. M. Lattimer, and E. F. Brown, *Astrophys. J.* **722**, 33 (2010).
- [25] K. Iida and K. Sato, *Astrophys. J.* **477**, 294 (1997).
- [26] D.G. Ravenhall and C.J. Pethick, *Astrophys. J.* **424**, 846 (1994).
- [27] T. Strohmayer, H. M. van Horn, S. Ogata, H. Iyetomi, and S. Ichimaru, *Astrophys. J.* **375**, 679 (1991).
- [28] S. Ogata and S. Ichimaru, *Phys. Rev. A* **42**, 4867 (1990).
- [29] C.J. Pethick and A. Y. Potekhin, *Phys. Lett. B* **427**, 7 (1998).
- [30] N. K. Johnson-McDaniel and B.J. Owen, [arXiv:1110.4650](https://arxiv.org/abs/1110.4650).
- [31] B.L. Schumaker and K.S. Thorne, *Mon. Not. R. Astron. Soc.* **203**, 457 (1983).
- [32] C.J. Pethick, N. Chamel, and S. Reddy, *Prog. Theor. Phys. Suppl.* **186**, 9 (2010).
- [33] H. Sotani, K. Nakazato, K. Iida, and K. Oyamatsu (unpublished).
- [34] See, e.g., B. A. Li, C. W. Chen, and C. M. Ko, *Phys. Rep.* **464**, 113 (2008); M. B. Tsang *et al.*, *Phys. Rev. Lett.* **102**, 122701 (2009).