Pechen and Tannor Reply: The authors of the Comment [1] do not criticize the rigorous results of our Letter [2]. They object to the conclusion that these results are "contrary to recent claims in the literature" and "can have profound implications for both theoretical and experimental quantum control studies."

One must distinguish between formal mathematical proof and induction from numerical experiments. Reference [3] claims to provide a formal proof that there are no traps in quantum control landscapes; no assumptions are made other than complete controllability. In [4], an additional assumption was introduced but it was portrayed as mild. Close inspection of this additional assumption shows that in at least one sense it is not mild—it is essentially equivalent to what needs to be proved, as we now explain.

Traps are defined through the condition $\delta J/\delta\varepsilon = 0$. The strategy of [3,4] is to use the chain rule

$$\frac{\delta J}{\delta\varepsilon} = \frac{\delta J}{\delta U}\frac{\delta U}{\delta\varepsilon},$$

where U is the unitary transformation produced by the pulse $\varepsilon(t)$. This equation decomposes the landscape analysis into a kinematic factor, $\delta J/\delta U$, and a dynamic factor, $\delta U/\delta\varepsilon$. As noted in [4], the kinematic factor was studied by von Neumann [5] (see also [6]) and shown to have no traps. But it is the dynamic landscape that is at question, and for this the behavior of the additional factor $\delta U/\delta\varepsilon$ is crucial. Reference [4] correctly points out that if the latter quantity is assumed to have full rank (also called "locally surjective" or "nonsingular") then $\delta J/\delta\varepsilon$ has no traps. To appreciate the significance of this assumption, imagine that U has only one element. Then the full rank condition is the condition that $\delta U/\delta\varepsilon \neq 0$, i.e., U cannot have a stationary point with respect to ε and, in particular, it cannot have a trap. Thus, hidden in the "mild" assumption that $\delta U/\delta\varepsilon$ is full rank is the assumption that the landscape of $U[\varepsilon]$ has no traps [7]. Away from the kinematic fixed points ($\delta J/\delta U = 0$) there is not even an attempt in the literature to provide a formal proof of the absence of traps.

References [4,8] assert that even if there are points where $\delta U/\delta\varepsilon = 0$, these points are "singular," "isolated," and "measure zero." This certainly sounds like these points are rare, but these same adjectives apply to the very traps we are looking for. These are not pathological conditions: they are the characteristics of standard maxima and minima. As to the argument that because the traps are of measure zero they will have a negligible effect on the landscape exploration, consider Fig. 1. It is seen that even a single trap can lead to large problems in exploring the landscape if it has a large attracting domain.

The Comment claims that the examples of second-order traps we provided in [2] are unphysical. Yet the systems and control objectives are precisely of the form considered in prior control landscape work. We considered the 3-level

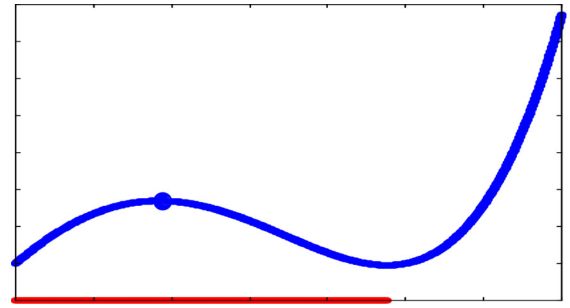

FIG. 1 (color online). A landscape with only one trapping point (blue circle) but with a large attracting domain (red interval). A local search over this landscape will be trapped for the majority of initial conditions.

Λ-system and generalizations thereof that appear in many branches of physics. The target observables were Hermitian operators, e.g., the difference of two final populations. The trap at $\varepsilon(t) = 0$ may look peculiar at first glance but it is not: a local search optimization that starts with an arbitrary temporal profile and sufficiently weak amplitude will get trapped in the $\varepsilon(t) = 0$ basin. There is no reason to think these are the only examples; e.g., Ref. [9] provided another example of a second-order trap.

The Comment also claims that to find the trap at $\varepsilon = 0$ one has to deliberately start with a weak field. The quantity that determines whether the amplitude is weak is the ratio $c_0 = \Omega_R/\omega$ between a typical Rabi frequency of the field Ω_R and a typical system transition frequency ω. In [10] we report a significant slow down of the search for the Λ-system already at $c_0 = 0.1$–0.2. For a transition dipole $\mu = 1$ a.u.$\approx 10^{-29}$ C m and $\omega = 5\times 10^{15}$ rad/s this corresponds to an irradiance of $I \approx 10^{12}$–10^{13} W/cm^2 [11], generally considered a strong field.

Finally, we address the impact that traps are likely to have on quantum control search algorithms. We were careful in our paper to say that we had found only second-order traps and that "[m]ore research will be required to establish if these points are true traps, but for the local search algorithms currently in use second-order traps pose virtually all the same numerical and experimental difficulties as true traps." Despite the extensive numerical tests described in the Comment, the question needs to be asked: to what extent do these runs span the full space of quantum control possibilities? We believe this question is still wide open. Given the lack of a formal mathematical proof of the absence of traps, and on the contrary, an example of a near-ubiquitous family of second-order traps, we stand fully by the conclusions of our paper that "the previous claims of the absence of traps, which were based on [the full rank] assumption, have to be completely rethought."

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