Spontaneous Breaking of the Spatial Homogeneity Symmetry in Wave Turbulence

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We report a surprising new result for wave turbulence which may have broader ramifications for general turbulence theories. Spatial homogeneity, the symmetry property that all statistical moments are functions only of the relative geometry of any configuration of points, can be spontaneously broken by the instability of the finite flux Kolmogorov-Zakharov spectrum in certain (usually one dimensional) situations. As a result, the nature of the statistical attractor changes dramatically, from a sea of resonantly interacting dispersive waves to an ensemble of coherent radiating pulses.

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Introduction.—Spatial homogeneity, the assumption that the property of translational symmetry, broken by each member of a statistical ensemble of fields, is restored on the average is one of the cornerstones of all turbulence theories [1]. It has been used broadly since Taylor formulated the theory of homogeneous turbulence and is seldom questioned. It is an extraordinarily convenient assumption for analysis. It means that statistical moments $\langle u(\mathbf{x})u(\mathbf{x}+\mathbf{r})u(\mathbf{x}+\mathbf{r}')\ldots\rangle$ of the fields $u(\mathbf{x}, t)$ are functions only of the relative geometry $\mathbf{r}$, $\mathbf{r}'$, ... Moreover, it ensures that the generalized Fourier transforms $A_{\mathbf{k}}(t) = (2\pi)^{-d}\int_{-\infty}^{\infty} u(\mathbf{x}, t)\exp(-i\mathbf{k}\cdot\mathbf{x})d\mathbf{x}$ are Dirac delta correlated (d is the dimension). For fields of mean zero, it says that the pair correlation is

$$\langle A_{\mathbf{k}}(t)A^*_{\mathbf{k}'}(t)\rangle = \delta(\mathbf{k}-\mathbf{k}')n_{\mathbf{k}}, \tag{1}$$

where the wave action density $n_{\mathbf{k}}$ is the Fourier transform of the two point average $\langle u(\mathbf{x})u^*(\mathbf{x}+\mathbf{r})\rangle$ and $\delta(\mathbf{k}-\mathbf{k}')$ is the Dirac delta function.

The new result of this Letter is that the spatial homogeneity symmetry can be broken spontaneously by an instability that broadens the correlation (1). We will derive this result for wave turbulence, the turbulence of weakly interacting dispersive waves. We consider equations of motion

$$i\frac{\partial u}{\partial t} = Lu + \lambda u^2 u^* \tag{2}$$

for a complex amplitude $u(\mathbf{x}, t)$ in one and in two spatial dimensions. The linear operator L is defined by $Le^{i\mathbf{k}\cdot\mathbf{x}} = \omega_k e^{i\mathbf{k}\cdot\mathbf{x}}$ with the eigenvalue $\omega_k = \sqrt{gk}$, $k = |\mathbf{k}|$, corresponding to the dispersion of surface gravity waves [2–5]. λ is a constant. The dynamics of (2) depends crucially on the sign of λ. This is shown in the numerical experiment of FIG 1, where Eq. (2) for one spatial dimension with external driving and damping is integrated in time. The simulation starts with wave turbulence (small amplitude, broad spectrum) initial conditions, and the sign of the nonlinearity is $\lambda = -1$. After an integration over 1000 time units, the sign of the nonlinearity is switched to $\lambda = 1$, and the integration is continued for another 1000 time units. The statistical properties for $t \leq 1000$, $\lambda = -1$ follow the predictions of wave turbulence theory, in particular, the Kolmogorov-Zakharov (KZ) steady state spectrum is achieved. For $t > 1000$, $\lambda = 1$ we observe that wave trains with high amplitude merge into coherent objects traveling with the group velocity. We will now show that switching the sign of λ leads to an instability that breaks the spatial homogeneity symmetry.

Unlike the case of general turbulence for which there is no consistent closure for the hierarchy of cumulant equations, wave turbulence has a natural asymptotic closure [6]. All statistical quantities, the spectral energy density, the long time behaviors of all higher cumulants, the structure functions, can be calculated from the solution of a single closed equation [7] for the isotropic wave action density $n_{\mathbf{k}}$. The kinetic equation is

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = T[n_{\mathbf{k}}] = \sum_{r=1} T_{2r}[n_{\mathbf{k}}], \tag{3}$$

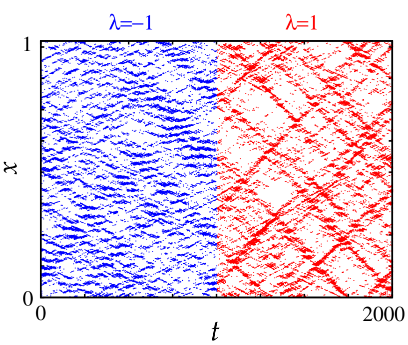

FIG. 1 (color online). Contour plot of points $|u(x,t)|^2 > 0.035$ in one spatial dimension with periodic boundary conditions for Eq. (2) with damping and driving. Damping is applied to modes both at very low and at high k, driving is applied to modes with wave numbers above the low-k damping. The sign of the nonlinearity is $\lambda = -1$ for $t \leq 1000$, and $\lambda = 1$ for $t > 1000$.

where $T[n_{\mathbf{k}}]$ is an asymptotic expansion capturing all the relevant physics of wave resonant processes [6–8]. For the equation of motion (2) its leading term is

$$T_4[n_{\mathbf{k}}] = 4\pi\lambda^2 \int n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \times \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}}\right) \times \delta(\omega + \omega_1 - \omega_2 - \omega_3) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}. \tag{4}$$

Added to (3) is an equation for the nonlinear renormalization of the frequency

$$\tilde{\omega}_k = \omega_k + 2\lambda \int n_{\mathbf{k}} d\mathbf{k}, \tag{5}$$

where the integral extends $\mathbf{k}$ space. A key observation is that whereas the kinetic equation is the same for both signs of λ, the frequency modification is not. The finite flux KZ solution, an exact solution of (4) is

$$n_{\mathbf{k}} = n_0(k) = DP^{1/3} k^{-d}, \tag{6}$$

where d is the dimension of $\mathbf{k}$, P is the constant energy flux and D a dimensional constant.

The remarkable new result is: We will show that if $\omega_k = \sqrt{gk}$, $d = 1$ and $\lambda = 1$, the KZ solution (6) is unstable to a perturbation which breaks the spatial homogeneity symmetry. Such perturbations are introduced into the theory by allowing the Fourier transform

$$n_{\mathbf{k}}(\mathbf{x}, t) = \int \langle u(\mathbf{x}, t) u^*(\mathbf{x} + \mathbf{r}, t)\rangle e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \tag{7}$$

of the pair correlator to depend weakly on the base coordinate $\mathbf{x}$. This broadened wave action density then satisfies a slightly revised kinetic equation with a Vlasov-like left-hand side

$$\frac{\partial n_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}} \tilde{\omega}_k \cdot \nabla_{\mathbf{x}} n_{\mathbf{k}} - \nabla_{\mathbf{x}} \tilde{\omega}_k \cdot \nabla_{\mathbf{k}} n_{\mathbf{k}} = \sum_{r=1} T_{2r}[n_{\mathbf{k}}], \tag{8}$$

where $\tilde{\omega}_k(\mathbf{x}, t) = \omega_k + 2\lambda \int n_{\mathbf{k}}(\mathbf{x}, t) d\mathbf{k}$. The Vlasov addition is nonlinear by virtue of the renormalized frequency $\tilde{\omega}_k$, and it vanishes for homogeneous ($\mathbf{x}$-independent) states. The space-independent solutions of (8) correspond to solutions of the kinetic Eq. (3), such as the KZ solution $n_0(k) = DP^{1/3}k^{-d}$.

Analysis.—We study the stability of homogeneous isotropic states $n_0(k)$ by adding small space-dependent perturbations $n_{\mathbf{k}}(\mathbf{x}, t) = n_0(k) + \delta n_{\mathbf{k}}(\mathbf{x}, t)$ with $\delta n_{\mathbf{k}}(\mathbf{x}, t) = \Delta(\mathbf{k}) \exp(i\mathbf{K} \cdot \mathbf{x} - i\Omega t)$ where $\mathbf{K}$ is the wave vector and Ω is the phase frequency of the modulation. The calculation parallels that of the analysis in [9] which examines short wave—long-wave coupling in the context of optical bump on tail instability. We assume, consistent with previous analyses of Balk and Zakharov [10] that $n_0(k)$ is stable with respect to spatially independent perturbations. Inserting $n_0(k) + \Delta(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x} - i\Omega t)$ in (8) yields

$$i\Delta(\mathbf{k})(\omega_k' \hat{\mathbf{k}} \cdot \hat{\mathbf{K}} - c)K - 2i\hat{\mathbf{k}} \cdot \hat{\mathbf{K}} K n_0'(k) \lambda \int \Delta(\mathbf{k}) d\mathbf{k} = \delta T_4, \tag{9}$$

where $K = |\mathbf{K}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $\hat{\mathbf{K}} = \mathbf{K}/K$, $n_0'(k) = dn_0/dk$. $c = \Omega/K$ is the phase velocity of the modulation and $\omega_k' = d\omega_k/dk$ is the group velocity of linear waves. Complex values of c and Ω will signal instability. δT_4 is a linear functional of $\Delta(\mathbf{k})$. To begin we ignore δT_4, and we will discuss its small influence later. Integrating over $\mathbf{k}$ we obtain

$$1 = 2\lambda \int \frac{n_0'(k) \hat{\mathbf{k}} \cdot \hat{\mathbf{K}}}{\omega_k' \hat{\mathbf{k}} \cdot \hat{\mathbf{K}} - c} d\mathbf{k}. \tag{10}$$

We begin with one spatial dimension $d = 1$. It is convenient to set $\hat{\mathbf{k}} \cdot \hat{\mathbf{K}} = 1$ and allow k to be positive or negative with $\omega_k = \sqrt{g|k|}$. Integration by parts yields

$$1 = 2\lambda \int_{-\infty}^{\infty} \frac{n_0(k) \omega_k''}{(\omega_k' - c)^2} dk. \tag{11}$$

We analyze two distributions $n_0(k)$. The first is $n_0(k) = N_0 \delta(k - k_0)$. Equation (11) yields $(c - \omega_{k_0}')^2 = 2\lambda N_0 \omega_{k_0}''$. Stability depends on the sign of $\lambda \omega_{k_0}''$ which is $-\mathrm{sgn}\lambda$ for $\omega_k = \sqrt{g|k|}$. Thus for $\lambda > 0$, c is complex and the monochromatic solution is modulationally unstable. This result is qualitatively similar to the Benjamin-Feir-Lighthill criterion for monochromatic waves. Next we take a KZ spectrum with a spectral maximum at k_0, $n_0(k) = DP^{1/3}|k|^{-1}$ for $|k| > k_0$ and $n_0(k) = 0$ for $|k| < k_0$. To keep the total wave action finite we can cut off the spectrum ($n_0(k) = 0$) also at high $|k|$, which corresponds to the effect of viscous damping. Writing the right-hand side of (11) as two integrals, over (k_0, ∞) and $(-\infty, -k_0)$, setting $k \to -k$ in the second, and making the transformations $k = \omega^2/g$, $\omega = \omega_0 \sigma$, $\omega_0 = \sqrt{gk_0}$, gives

$$1 + \alpha \int_1^{\infty} \frac{z^2}{\sigma^2} \left(\frac{1}{(z - \sigma)^2} + \frac{1}{(z + \sigma)^2}\right) d\sigma = 0. \tag{12}$$

The dimensionless parameter $z = g/(2c\omega_0) = \omega_0'/c$ is the ratio of group velocity at the spectral peak and the phase velocity of the perturbation. A complex z indicates an instability. The dimensionless parameter $\alpha = 4\lambda DP^{1/3}/\omega_0$, $|\alpha| \ll 1$, the nondimensional energy flux, essentially measures the weakness of the turbulence. Its smallness is also important to guarantee (see [8]) that the ratio of the linear and the nonlinear time scale $t_L/t_{NL} = \omega_k^{-1}/(n_k/\frac{dn_k}{dt})$, which is a function of the frequency ω and is equal to the product of α^2 and ω_0^2/ω^2, remains small throughout the range, ω_0 to ∞, over which the KZ spectrum obtains. That ensures [8] that the wave turbulence closure remains valid on the KZ solution of the kinetic equation.

Integrating (12) gives

$$1 + \alpha\left(2 + \frac{2}{z}\ln\frac{1-z}{1+z} + \frac{1}{1-z} + \frac{1}{1+z}\right) = 0. \tag{13}$$

For α small, the root z is close to unity and the leading order of (13) is $1 + \alpha/(1-z) \approx 0$ and the root is $z \approx 1 + \alpha$. This shows that the phase speed of the disturbance almost matches the group speed of the wave packet at the spectral peak k_0. For λ and therefore α negative, the root $z \approx 1 + \alpha$ of (13) is real and less than unity. For λ and thus α positive, we have a root $|z| > 1$. The appropriate branches of the logarithm are now $\ln(1-z) \approx \ln|1-z| \pm i\pi$. Including the leading order imaginary term we obtain $1 + \alpha(\pm 2i\pi + 1/(1-z)) \approx 0$, so the root for α small is $z \approx 1 + \alpha \mp 2i\pi\alpha^2$. The wave speed $c = \Omega/K = \omega_0'/z$ is complex and the solution $n_0(k)$ unstable. The instability growth rate is of order α^2. Interestingly, this matches the inverse time scale for the spectrum to relax to the KZ state.

This shows that a small spatial inhomogeneity can be enhanced by the interaction with the almost random waves of homogeneous weak turbulence. Landau damping and amplification of plasma oscillations provides a useful analogue for the situation here. The ensemble of wave packets with density $n_0(k)$ plays the role of the electron distribution whereas $\exp(i\mathbf{K}\cdot\mathbf{x} - i\Omega t)$ is the perturbing wave. The perturbing wave is amplified (damped) if the wave number of the spectral peak lies to the right (left) of that of the perturbing wavetrain for which cases $\mathrm{Re}(\omega_k'/c)$ is greater (less) than unity.

In two dimensions $d = 2$, we take the KZ distribution $n_0(k) = DP^{1/3}k^{-2}H(k - k_0)$, where H is the Heaviside function and $k = |\mathbf{k}|$. Then $n_0'(k) = DP^{1/3}k^{-2}\delta(k - k_0) - 2DP^{1/3}k^{-3}H(k - k_0)$ in (10). Integrating over the angular coordinate ϕ, $\cos\phi = \hat{\mathbf{k}}\cdot\hat{\mathbf{K}}$ and the variable $\omega' = \sqrt{g/k}/2$ gives

$$\frac{1}{8\pi\alpha} = \frac{1}{z}\sin^{-1}z - \frac{1}{4}\frac{1}{\sqrt{1-z^2}} - \frac{3}{4} \tag{14}$$

for $z = \omega_0'/c$, or with $z = \sin(\zeta)$,

$$\frac{1}{8\pi\alpha} = \zeta\,\mathrm{cosec}(\zeta) - \frac{1}{4}\sec(\zeta) - \frac{3}{4}. \tag{15}$$

For $\alpha \ll 1$, the roots of (14) and (15) are real for both signs of α. To leading order, the root is $\pi/2 + 2\pi\alpha$. The perturbing wave train is neutrally stable. This shows that a spatial inhomogeneity is not increased by the interaction of waves in two dimensional weak turbulence.

Finally, we discuss the small correction due to δT_4 on the right-hand side of (13). Writing $n_{\mathbf{k}}(\mathbf{x}, t) = n_0(k) + \Delta(\mathbf{k})\exp(i\mathbf{K}\cdot\mathbf{x} - i\Omega t)$ in $T_4[n_k]$ and obtaining, after linearization in Δ,

$$\begin{aligned}\delta T_4 = 4\pi\lambda^2 \int &[\{n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}_1}n_{\mathbf{k}_3}\}\Delta(\mathbf{k}) \\ &+ P^{123}\{n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}}n_{\mathbf{k}_2} - n_{\mathbf{k}}n_{\mathbf{k}_3}\}\Delta(\mathbf{k}_1)] \\ &\times \delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ &\times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)d\mathbf{k}_{123}.\end{aligned}$$

The upper indices of P^{123} denote the cyclic permutations over 1, 2, 3. We compute this term for the one dimensional case. Dividing (9) by $i(\omega_k' - c)K$, integrating over k, dividing by $\int_{-\infty}^{\infty}\Delta(k)dk$ and nondimensionalizing all wave numbers with k_0 and frequencies with ω_0 (note $\omega_k' k = \omega_k/2$) gives $-\pi i\alpha^2 I/2$ on the right-hand side of (13). I is a real dimensionless integral. Including this term we obtain a higher order correction $\pi\alpha^3 I/2$ in $\mathrm{Im}(z)$ of the unstable mode $\exp(i\mathbf{K}\cdot\mathbf{x} - i\Omega t)$. The collision term has therefore only a small effect on the instability, but eventually it will erode the correlation that evolves from the instability.

Simulations.—The instability that we have described is a statistical property of an ensemble of trajectories, whereas the behavior shown in FIG. 1 is the manifestation of this instability for a single trajectory. We now explore the instability by following an ensemble of trajectories numerically. First an ensemble of 400 000 initial conditions is created. These initial states are random KZ-distributed waves with an additional common small spatial inhomogeneity. Using this ensemble of initial conditions, the equation of motion with $\lambda = 1$ is integrated over 70 time units and, for comparison, the equation with $\lambda = -1$ is integrated starting from the same initial conditions. No external driving and damping is applied in this experiment.

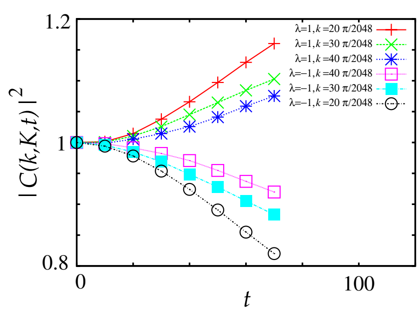

FIG. 2 (color online). Time evolution of the correlation $|C(k, K, t)|^2 = |\langle A_k(t)A_{k+K}^*(t)\rangle|^2/|\langle A_k(0)A_{k+K}^*(0)\rangle|^2$ with $K = 6\pi/2048$ for an ensemble of 400 000 trajectories for the Eq. (2) with $\lambda = 1$ and with $\lambda = -1$. The initial conditions are Kolmogorov-Zakharov distributed $n_k \sim k^{-1}$ with a Gaussian amplitude distribution and random phases for $|k| \geq 20\pi/2048$ and $n_k = 0$ for $|k| < 20\pi/2048$ (the wave number space is $-\pi < k \leq \pi$). A small spatial modulation is superimposed on these random initial conditions, and this modulation is the same for each member of the ensemble. The system is not externally damped or driven. This correlation grows for $\lambda = 1$, reflecting an instability of wave turbulence against spatially inhomogeneous perturbations. There is no such instability for $\lambda = -1$, and the correlation decays.

Figure 2 shows time evolution of the ensemble average $|\langle A_k(t)A^*_{k+K}(t)\rangle|^2$ for various values of k with the same modulation K of the initial conditions. We observe that $|\langle A_k(t)A^*_{k+K}(t)\rangle|^2$ grows in time for $\lambda = 1$, indicating the amplification of the small initial spatial inhomogeneity. This shows that the nonequilibrium state of wave turbulence is not a statistical attractor in this case. In contrast, this correlation decays for $\lambda = -1$, which shows that homogeneous wave turbulence is attractive and small spatial inhomogeneities vanish. This decay effect can be attributed to the nonlinear terms in (8). In particular, the collision term will, via four wave interactions, remove energy from the perturbing waves. Longer simulations for the case $\lambda = 1$ show that the correlation eventually decays after the initial surge. Again, we attribute this to the nonlinearities in (8). The system eventually settles down at a statistical attractor which inherits the spatial homogeneity symmetry of the equation of motion. This new state becomes visible in Fig. 1 at high t, where moving localized structures appear. We have studied these coherent structures and their influence on the statistical properties of turbulence in [5].

*Conclusions.—*We have discovered an instability that breaks the spatial homogeneity of wave turbulence by enhancing small long-wave modulations. In this process, the random waves of wave turbulence feed energy into spatial modulations. We have shown this using a linear stability analysis of the statistical state of homogeneous wave turbulence. This analysis is based on the generic kinetic equation with a Vlasov term (8). The results are therefore expected to apply to many nonlinear wave systems and not just to the model (2). Our findings are confirmed by numerical simulations of an ensemble of trajectories of Eq. (2) for wave turbulence initial conditions with a small spatial inhomogeneity.

The instability occurs in (2) for the same sign of the nonlinearity ($\lambda = 1$) for which monochromatic waves are Benjamin-Feir unstable. In fact, the Benjamin-Feir-Lighthill criterion follows from our analysis for a delta-distributed spectrum. However, the breaking of spatial homogeneity for a Kolmogorov-Zakharov spectrum is a statistical process that is obtained by averaging over many interactions of waves that decrease or increase spatial inhomogeneity. Wave turbulence is neutrally stable for $\lambda = -1$. We suppose that the Benjamin-Feir-Lighthill criterion is a necessary condition for this instability.

The outcome crucially depends on the dimension: Remarkably, isotropic wave turbulence in two dimensions is more robust in that it is not subject to this instability. The random waves transfer no energy to the inhomogeneity in two dimensions, and the perturbation is neutrally stable. Quasi-one-dimensional wave systems are apparently the most vulnerable and likely candidates for this instability.

The nonequilibrium state that finally emerges from this instability is radically different from wave turbulence. As we have shown in [5] the system is then governed by radiating pulses that lead to a spectrum that is steeper than the Kolmogorov-Zakharov spectrum of wave turbulence. Examples for the spontaneous formation of coherent structures in turbulence are rogue waves in the ocean [11] and in nonlinear optics [12], and solitons in nonlocal nonlinear media [13]. Radiating solitons have previously been discussed in nonlinear optics [14]. We emphasize the point that whereas spatial homogeneity is broken, eventually that property is restored when there are enough coherent pulses in the system. It is fascinating that the breaking of spatial homogeneity is the means through which the system leaves one unstable fixed point (the KZ solution) and reaches the stable attractor of an ensemble of radiating pulses [5].

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