Quantum Dynamics of a Bose Superfluid Vortex

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We derive a fully quantum-mechanical equation of motion for a vortex in a 2-dimensional Bose superfluid in the temperature regime where the normal fluid density $\rho_n(T)$ is small. The coupling between the vortex "zero mode" and the quasiparticles has no term linear in the quasiparticle variables—the lowest-order coupling is quadratic. We find that as a function of the dimensionless frequency $\tilde{\Omega} = \hbar\Omega/k_BT$, the standard Hall-Vinen-Iordanskii equations are valid when $\tilde{\Omega} \ll 1$ (the "classical regime"), but elsewhere, the equations of motion become highly retarded, with significant experimental implications when $\tilde{\Omega} \gtrsim 1$.

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Quantum vortices were first predicted in ^{4}He superfluid by Onsager [1] and found experimentally a decade later [2]. Vortices, along with quasiparticles, constitute the two basic excitations in many condensed matter systems [3]; they may also have existed as topological defects in the early universe [4].

Remarkably, the fundamental question of how vortices and quasiparticles interact, and how vortices move is very controversial, notably for superfluids [5–7]. The argument is usually phrased in terms of forces acting on a vortex moving with respect to both the superfluid [having local velocity $\boldsymbol{v}_s(\mathbf{r})$ and superfluid density ρ_s] and the normal fluid [with velocity $\boldsymbol{v}_n(\mathbf{r})$ and density ρ_n]. Then, the classical equation of motion for the semiclassical vortex coordinate $\boldsymbol{R}_v(t)$ (here taken to be a point in the plane—we discuss the 3D problem at the end) is usually written [8] as

$$M_v\ddot{\mathbf{R}}_v - \boldsymbol{f}_M - \boldsymbol{f}_{qp} - \mathbf{F}_{ac}(t) = 0, \tag{1}$$

where $\mathbf{F}_{ac}(t)$ is some driving force, M_v is the vortex mass, $\boldsymbol{f}_M = \rho_s\boldsymbol{\kappa} \times (\dot{\mathbf{R}}_v - \boldsymbol{v}_s)$ is the (uncontroversial) Magnus force for a vortex with circulation $\boldsymbol{\kappa} = \hat{\mathbf{z}}h/m$, and the quasiparticle force is

$$\boldsymbol{f}_{qp} = D_0(\boldsymbol{v}_n - \dot{\mathbf{R}}_v) + D_0'\hat{\mathbf{z}} \times (\boldsymbol{v}_n - \dot{\mathbf{R}}_v), \tag{2}$$

where $D_0(T), D_0'(T)$ depend strongly on the temperature T. The classic discussion of Iordanskii [9] yields

$$D_0'(T) = -\kappa\rho_n(T). \tag{3}$$

These "HVI equations," due to Hall, Vinen, and Iordanskii [8,9], have been used to analyze thousands of experiments in superfluids and superconductors in the last 60 years [10]. However, there is no consensus on the value of either the mass M_v (estimates in the literature range from zero to infinity [7]) or of the coefficients $D_0(T), D_0'(T)$. Indeed, Thouless *et al.* [5] find $D_0'(T) = 0$ for all T, and scattering analyses give various results for $D_0'(T)$ [6,9,11–13]. The resolution of these questions has become an important unsolved problem in physics. We briefly discuss the experimental situation below.

Previous analyses have been restricted to a local (in space and time) description, derived from semiclassical or perturbative calculations of quasiparticle scattering off a static vortex, with no vortex recoil [6,9,11–13]. This yields forces acting instantaneously on a quasiclassical vortex coordinate $\mathbf{R}_v(t)$. There is no general agreement between these calculations (which are rendered difficult by the long-range vortex profile). Our tactic has been to formulate the problem fully quantum mechanically, in terms of an equation of motion for the vortex reduced density matrix $\bar{\boldsymbol{\rho}}(\mathbf{R}, \mathbf{R}'; t)$, which is obtained by integrating out all quasiparticle degrees of freedom. Here the vortex coordinate states $|\mathbf{R}\rangle$, $|\mathbf{R}'\rangle$ are defined by the position of the vortex node appearing in the many-body wave function. We then define a vortex "center of mass" coordinate $\mathbf{R}_v = \frac{1}{2}(\mathbf{R} + \mathbf{R}')$, and a "quantum fluctuation coordinate" $\boldsymbol{\xi} = \mathbf{R} - \mathbf{R}'$. Equations of motion are then found for these two quantum variables, with the vortex recoil automatically incorporated. Remarkably, in thermal equilibrium we find that the results largely depend on one key parameter, the ratio $\tilde{\Omega} = \hbar\Omega/k_BT$, where Ω is the characteristic frequency of the vortex motion, and k_BT is the thermal energy of the quasiparticles. When $\tilde{\Omega} \ll 1$ we are in a "classical regime," where we find that the HVI equations (1)–(3) can be justified, with the addition of a nontrivial fluctuation force on the right-hand side of (1). However, when $\tilde{\Omega} \gg 1$ we are in a quantum regime which has seen little experimental exploration, and where the vortex equations of motion are rather different.

Two key features of the analysis [14] are as follows: (i) The widespread assumption of a vortex-quasiparticle coupling which is linear in the quasiparticle variables is not correct. The vortex is a solitonic excitation of the same field as the quasiparticle excitations. Linear couplings are then forbidden: for the vortex to be a *bona fide* minimum action solution to the equations of motion, the lowest-order coupling has to be at least quadratic in all fluctuation variables [15]. One then needs to find a "renormalized" coupling to new fluctuation variables that are correctly orthogonalized to each other and to the vortex "zero mode"; this turns out to be very complicated [14]. The renormalized coupling, which is singular at low momentum transfer, is indeed quadratic in these new variables. (ii) Integration over quasiparticle coordinates then produces time-retarded, long-range interactions between different points on a vortex "worldline." A nonperturbative path integral treatment (required to deal with the singular vortex-quasiparticle interaction) then yields extra "memory forces" in the equations of motion, which become important in the quantum regime $\tilde{\Omega} > 1$.

(i) Results.—At low temperatures, Bose liquids are described by an effective Hamiltonian of the form [16]:

$$H = \int d^2r \left(\frac{\rho}{2m_0^2} (\hbar \nabla \Phi)^2 + \epsilon[\eta, \nabla \eta] \right), \tag{4}$$

with density $\rho = \rho_s + \eta(\mathbf{r})$, density fluctuations $\eta(\mathbf{r})$, an energy functional $\epsilon[\eta, \nabla\eta]$ whose form depends on which superfluid we study, and a superfluid phase $\Phi(\mathbf{r})$. This Hamiltonian is restricted to length scales $\gg a_0 = \hbar/m_0 c_0$, to energies $\ll m_0 c_0^2$, and to velocities $\ll c_0$, the sound velocity, where $c_0^2 = \rho_s (d^2\epsilon/d\eta^2)|_{\eta=0}$.

We emphasize the limitations of this hydrodynamic formulation. It is valid for both low-density Bose gases and dense superfluids like ^{4}He, provided $k_B T \ll m_0 c_o^2$ (so that $\rho_n \ll \rho$; in, e.g., ^{4}He superfluid, this means $T \lesssim 0.7$ K), and likewise for perturbations of frequency $\Omega \ll m_0 c_0^2/\hbar$. With these restrictions it is valid for arbitrary ratios of the "crossover parameter" $\tilde{\Omega} = \hbar\Omega/k_B T$. However, it does not include interquasiparticle interactions, which are very small in this low-T regime and have no bearing on the form of the quasiparticle-vortex interaction, but which are essential for the macroscopic transport of energy and momentum [17]. Nor do we study here the role of the boundaries—such geometric effects are crucial in understanding the vortex mass [7], although they hardly affect the vortex-quasiparticle interaction.

(a) Equations of Motion.—The results are more transparent when Fourier transformed. Defining $\mathbf{R}_v(t) = \int_0^\infty d\Omega \mathbf{R}_v(\Omega) e^{i\Omega t}$, we write the equation of motion as

$$R_i^v(\Omega) = A_{ij}(\Omega, n_\mathbf{q}) F_j(\Omega), \tag{5}$$

where $n_\mathbf{q}$ is the quasiparticle distribution over momentum $\mathbf{q}$, and the total "driving force"

$$\mathbf{F} = \mathbf{F}_{ac}(\Omega) - q_v \boldsymbol{\kappa} \times \mathbf{J}(\Omega) + \mathbf{F}^{\text{fluc}}(\Omega, n_\mathbf{q}) \tag{6}$$

sums an external driving field, an internal local transverse force $\boldsymbol{f}_\perp = -q_v \boldsymbol{\kappa} \times \mathbf{J}(\Omega)$, where $\mathbf{J} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n$ is the total current, and a fluctuation term $\mathbf{F}^{\text{fluc}}(\Omega, n_\mathbf{q})$. The 2×2 "admittance matrix" $A_{ij} = A^\parallel \delta_{ij} + A_{ij}^\perp$, where

$$\begin{aligned} A^\parallel &= \mathbb{D}^{-1}[-\Omega^2 M_v(\Omega, n_\mathbf{q}) + i\Omega D_\parallel(\Omega, n_\mathbf{q})], \\ A^\perp &= \mathbb{D}^{-1}[\hat{\sigma}_y \kappa \rho \Omega - \hat{\sigma}_x |\Omega| d_\perp(\Omega, n_\mathbf{q})], \end{aligned} \tag{7}$$

where $\hat{\sigma}$ are the usual Pauli matrices, and

$$\mathbb{D}(\Omega, n_\mathbf{q}) = [\Omega^2 M_v - i\Omega D_\parallel]^2 - [\kappa\rho\Omega - i|\Omega| d_\perp]^2. \tag{8}$$

If we write $A_{ij} = \Omega \sigma_{ij}$, then $\sigma(\Omega, T)$ is analogous to a conductivity tensor, with $D_\parallel(\Omega)$ playing the role of the longitudinal resistivity.

The key difference between (5)–(8) and previous results for this problem lies in the frequency dependence of $D_\parallel(\Omega, n_\mathbf{q})$, $d_\perp(\Omega, n_\mathbf{q})$, $M_v(\Omega, n_\mathbf{q})$, and in the correlator $\chi_{ij}(\Omega, n_\mathbf{q}) = \langle F_i^{\text{fluc}}(\Omega, n_\mathbf{q}) F_j^{\text{fluc}}(-\Omega, n_\mathbf{q}) \rangle$ of the fluctuation force $\mathbf{F}^{\text{fluc}}$. Quite generally we find that when the quasiparticles are in equilibrium at temperature T, the "viscous" terms [i.e., $D_\parallel(\Omega, T)$, $d_\perp(\Omega, T)$, and $\chi_{ij}(\Omega, T)$] in (5)–(8) take the form $Q(\Omega, T) = f(T) g(\tilde{\Omega})$. The effective mass $M_v(\Omega, T)$ has a more complicated behavior.

(b) Quantum-Classical crossover.—Consider first $D_\parallel(\tilde{\Omega}, T)$, shown in Fig. 1. Over the whole range of $\tilde{\Omega}$ and T, $D_\parallel(\tilde{\Omega}, T) = D_0(T) g_D(\tilde{\Omega})$, where $D_0(T)$ is just the coefficient in (2), and where $g_D(\tilde{\Omega})$ decreases smoothly from $g_D(0) = 1$ in the classical limit to $g_D(\tilde{\Omega} \to \infty) = 1/16$ in the quantum limit [18]. The fluctuation correlator $\chi_{ij}(\tilde{\Omega}, T)$, shown in Fig. 2, is a little more complicated.

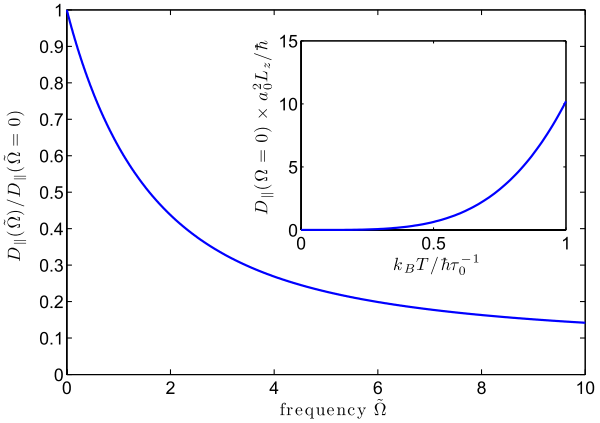

FIG. 1 (color online). The longitudinal damping coefficient $D_\parallel(\tilde{\Omega}, T)$. Main figure: The dependence of $D_\parallel(\tilde{\Omega}, T)$ on $\tilde{\Omega}$, normalized to its zero frequency value $D_0(T)$. Inset: The coefficient $D_0(T)$, proportional to T^4, plotted as a function of the dimensionless temperature $(\tau_0/\hbar) k_B T$.

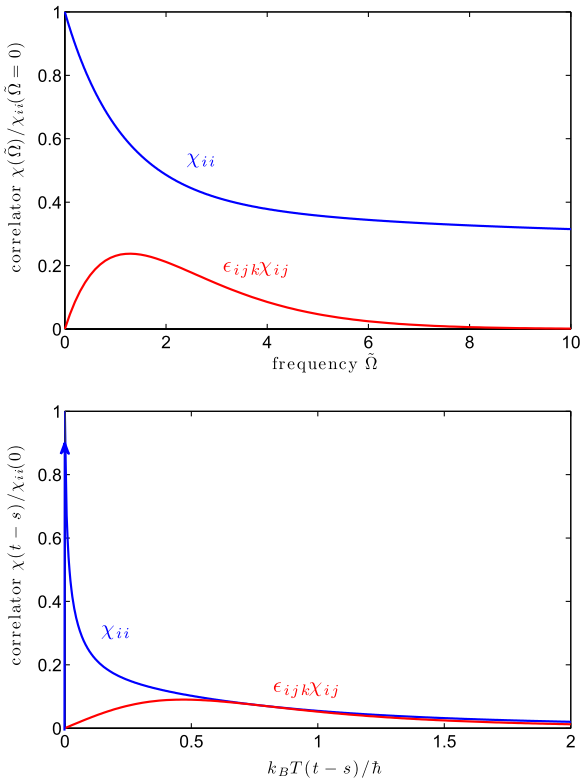

FIG. 2 (color online). The correlator χ_{ij} of the fluctuating force, plotted as a function of $\tilde{\Omega}$ (top) and the dimensionless time $k_BT(t-s)/\hbar$ (bottom). The plots are normalized to χ_{ii} at zero frequency (top) and at zero time (bottom). The arrow at $t-s=0$ in the bottom figure represents a δ-function contribution (see text).

Ordinarily, one expects the noise correlator in a quantum Langevin equation to have the form $\chi_{QL}(\tilde{\Omega},T)\sim f(T)\tilde{\Omega}\coth\frac{1}{2}\tilde{\Omega}$, i.e., a strictly increasing function of $\tilde{\Omega}$. However, Fig. 2(a) shows a quite different behavior: like $D_{\|}(\tilde{\Omega},T)$, the longitudinal correlator $\chi_{ii}(\tilde{\Omega},T)$ *decreases* smoothly with $\tilde{\Omega}$, now with the limiting behavior

$$\frac{\chi_{ii}(\tilde{\Omega})}{\chi_0^{\|}}\to\begin{cases}\left(1-\frac{\zeta(3)}{2\zeta(4)}\tilde{\Omega}\right) & (\tilde{\Omega}\to 0)\\ \frac{\zeta(5)}{4\zeta(4)}\left(1+\frac{5\zeta(6)}{\zeta(5)}\tilde{\Omega}^{-1}\right) & (\tilde{\Omega}\to\infty),\end{cases}\tag{9}$$

where $\chi_0^{\|}(T)=\chi_{ii}(\tilde{\Omega}=0,T)=AD_0(T)$, with $A=k_BT/L_z\pi$ (a relationship coming from the fluctuation-dissipation theorem). On the other hand, in both limits the transverse part $\epsilon_{ijk}\chi_{ij}$ of $\chi_{ij}(\Omega,T)$ is zero; it rises to a maximum value $\sim 0.2\chi_0^{\|}(T)$ in the crossover regime $\tilde{\Omega}\sim 1$.

A Fourier transform back to the time domain [see Fig. 2(b)] reveals an initial δ function in $\chi_{ii}(t-s,T)$ [because $\chi_{ii}(\tilde{\Omega},T)$ is everywhere finite and positive] followed by a slow decay $\propto(t-s)^{-2}$. Similar behavior is found for $D_{\|}(t-s,T)$ [14]. The transverse correlator rises from zero at zero time, and decays more rapidly [like $(t-s)^{-3}$] at long times.

The transverse function $d_{\perp}(\tilde{\Omega},T)$ also shows a characteristic crossover behavior—we do not elaborate here because $d_{\perp}(\tilde{\Omega},T)$ is always very small compared to $D_{\|}(\tilde{\Omega},T)$. Full details on all these functions are found in the Supplemental Material [14].

Finally, consider the vortex inertial mass $M_v(\Omega,T)$ appearing in (5)–(8). This is well-known to depend on the sample geometry [7]; for a circular container of radius $R_0\gg a_0$, in the $\Omega\to 0$ limit, we easily verify the well-known hydrodynamic result [19]

$$M_v(0)=\pi\rho_s a_0^2\left[\ln\left(\frac{R_0}{a_o}\right)+\gamma_E+1/4\right],\tag{10}$$

where γ_E is Euler's constant. Naively, one expects that in the quantum limit $\tilde{\Omega}\gg 1$, the $\ln(R_0/a_0)$ factor in (10) will be replaced by $\ln(c_0/a_0\Omega)$ (the length scale c_0/a_0 being set by the distance quasiparticles can travel in a time $\sim\Omega^{-1}$). However, the actual behavior is more subtle: there is a "radiation reaction" term $\propto d^3\mathbf{R}_v/dt^3$ in the equation of motion, analogous to that in electrodynamics, and to deal with this one must go beyond the expansion in powers of $\dot{\mathbf{R}}_v/c_0$ being used here. This problem lies outside the scope of the present paper [20].

(c) *Real-time dynamics.*—Remarkably, the results given above allow us to write simplified *local* equations of motion in *both* quantum and classical limits. In the classical limit, Fourier transforming back gives precisely the HVI equations (1)–(3), but with an added noise term:

$$M_v\ddot{\mathbf{R}}_v-\boldsymbol{f}_M-\boldsymbol{f}_{qp}-\mathbf{F}_{ac}(t)=\mathbf{F}_{\rm fluc}^{(cl)}(t),\tag{11}$$

where the classical noise force $\mathbf{F}_{\rm fluc}^{(cl)}(t)$ has the correlator

$$\chi_{ij}^{(cl)}(t-s,T)\sim\chi_o^{\|}(T)\delta_{ij}\delta(t-s),\tag{12}$$

i.e. an entirely longitudinal Markovian noise. However this equation is only meaningful on coarse-grained time scales $\gg\hbar/k_BT$; for shorter times, the time-retarded nature of the correlations becomes crucial, and as Fig. 2 makes clear, the fluctuation correlator $\chi_{ij}(t-s,T)$ then becomes anisotropic and highly non-Markovian, and the HVI equations simply do not apply.

In the opposite quantum limit $\tilde{\Omega}\gg 1$, one may again write a local equation like (11), of HVI form, again with an added noisy fluctuating force. However, now the coefficients are different; $D_0(T)$ in (2) is replaced by $D_0(T)/16$, and the quantum noise correlator $\chi_{ij}^{(Q)}(t-s,T)=\frac{\zeta(5)}{4\zeta(4)}\chi_{ij}^{(cl)}(t-s,T)$ (again entirely longitudinal). In this limit, valid for time scales $\ll\hbar/k_BT$ (but $\gg\hbar/m_0c_0^2$), the coefficients in these "quantum HVI" equations arise solely from the δ-function contributions to the correlation functions—all retarded parts are suppressed.

We would like to emphasize how unusual these results are in detail. It is quite remarkable to have 2 equations of exactly the same form (but quite different coefficients) in these two limits, but with a quite different form in the crossover between them; it is more illuminating to look at it in frequency space, as above. And yet, very surprisingly (at least to us), the Iordanskii force is quite unaffected by this—apart from the very small correction term $d_\perp(\tilde{\Omega}, T)$, the Iordanskii force is independent of frequency, and can be treated as entirely local, and (3) is reproduced exactly in our derivation (which is quite different from previous scattering theory calculations).

(ii) Experimental Implications.—The results above justify the HVI equations [8,9], and the phenomenology based on these [10], in the classical regime. However away from this regime we find a quite new phenomenology. Clean experimental tests of this will require (a) that the vortex not be coupled to some other object (e.g., charged ions), which change its natural dynamics, and (b) that the vortex be coupled to the natural excitations of the system (as opposed to, e.g., a source of external quasiparticles, which can couple linearly to the vortex). The results will also change in situations where the vortex is being ''dragged'' by some external time-varying potential [5], since such potentials may strongly distort the vortex in the region where they act.

The most obvious direct experimental realization of the results here would be in 2-dimensional Bose-condensed atomic gases; the dynamics of single vortices can then be tracked in experiments [21] (e.g., in their spiraling out from the trap center), and the viscous coefficients can then, in principle, be extracted from such measurements [22]. A detailed treatment using the present equations is quite lengthy, and will be presented elsewhere. Our results are also clearly relevant to experiments on turbulence in superfluid ^{4}He, which have recently begun to probe the quantum regime [23], and any theory of vortex tunneling in ^{4}He or cold gases must include the viscous effects described here (which are very far from being described by a simple Caldeira-Leggett coupling [24]). One difficulty with turbulence or tunneling is that in most experimental cases the vortices are 3-dimensional objects, and the vortex line may distort in many ways that are impossible to capture analytically. It would nevertheless be interesting to extend, at least numerically, the existing theories of quantum turbulence [25] and vortex tunneling [26] in ^{4}He to include the effects discussed here. Finally, we emphasize that the results given here are not applicable to fermionic superfluids like ^{3}He or superconductors—the form of the vortex-quasiparticle interaction is quite different in these systems.

To summarize: within the constraints of the low-T hydrodynamic picture, we find that the HVI equations can be justified in a purely quantum-mechanical treatment, with the addition of a fluctuation noise term, provided one is in the classical regime $\tilde{\Omega} \ll 1$. Outside this regime, one needs to use a more general set of equations, which show strong memory effects in the time domain.

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