

## Dynamics of a Qubit as a Classical Stochastic Process with Time-Correlated Noise: Minimal Measurement Invasiveness

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So far it has been shown that the quantum dynamics cannot be described as a classical Markov process unless the number of classical states is uncountably infinite. In this Letter, we present a stochastic model with time-correlated noise that exactly reproduces any unitary evolution of a qubit and requires just four classical states. The invasive updating of only 1 bit during a measurement accounts for the quantum violation of the Leggett-Garg inequalities. Unlike in a pilot-wave theory, the stochastic forces governing the jumps among the four states do not depend on the quantum state but only on the unitary evolution. This model is used to derive a local hidden variable model, augmented by 1 bit of classical communication, for simulating entangled Bell states.

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It is a well-established fact that the quantum dynamics among a finite set of mutually exclusive alternatives (like the *up* and *down* states of a  $1/2$  spin) cannot be reduced to a Markov process among a finite set of classical states. Indeed a time-homogeneous Markov process on a finite space always relaxes into a stationary probability distribution, as a consequence of the Perron-Frobenius theorem [1]. The impossibility of exactly simulating a quantum system through a Markov process on a finite classical space is also a consequence of a theorem proved in Ref. [2]. There we showed that the dimension of the classical space cannot be smaller than the quantum state manifold dimension under the hypothesis of Markov dynamics; that is, the classical space must be uncountably infinite. Pilot-wave theories provide an example of this overflow of classical resources. Indeed in their framework the dynamics among the set of alternatives explicitly depends on the quantum state, which actually turns to be part of the classical description. Pilot-wave theories of finite-dimensional quantum systems were considered, for example, in Refs. [3,4]. While they reintroduce a classical realistic picture of the quantum world, they are unavoidably characterized by a feature that is absent in prequantum physics, namely, the invasiveness of measurements. Thus, measurements do not provide a mere updating of knowledge about the actual state but intrinsically introduce a perturbation on the system. In every known classical model of quantum dynamics, this perturbation demands an invasive updating of an uncountably infinite amount of information. For example, in the case of a qubit, two continuous real variables need to be updated.

The Leggett-Garg inequalities provide a useful test for deciding if a set of data can be explained by a measurement-noninvasive classical theory [5]. Indeed they are violated by quantum mechanics. These inequalities are analogous to the Bell inequalities for the Einstein-Podolsky-Rosen experiment [6] with the local

measurements being replaced by two consecutive measurements on a two-state quantum oscillator. This analogy suggests an interesting question. On the one hand, the violation of the Leggett-Garg inequalities demands the invasiveness of measurements in any classical theory of two-state quantum oscillators. On the other hand, the violation of the Bell inequality implies that a classical simulation of Bell correlations requires some communication between the parties. It is known that a finite amount of classical communication, namely, 1 bit, can actually account for this violation [7]. Thus it is natural to wonder if it is possible to simulate the quantum dynamics by a classical model that needs the invasive updating of a finite amount of information.

Generalizing the case discussed in Ref. [5], in this Letter we consider the scenario where two consecutive measurements of the same observable are performed, at times  $t_0$  and  $t_1 > t_0$ , on a qubit undergoing a generic unitary evolution. The outcome of each measurement is one of two orthogonal states, denoted by  $|\pm 1\rangle$ . They can be, for example, the left and right states of a double-well system [5]. We show that this scenario can be classically simulated by a two-bit classical model with time-correlated noise. The key ingredients of our model are *time correlation* of the noise and *minimal measurement invasiveness*. The first ingredient is required for circumventing the constraint on the classical space dimension proved in Ref. [2], as this constraint does not hold for non-Markov processes. The second ingredient accounts for the invasiveness of measurements by demanding that only 1 bit is invasively updated. Unlike in a pilot-wave theory, the noise governing the dynamics is independent of the measurement times  $t_0$  and  $t_1$  and depends only on the unitary evolution. In particular, the noise is independent of the quantum state. This feature and the finite number of classical states distinguish our result from the previous ones, such as the class of hidden variable theories considered in Ref. [3], where

the stochastic matrices are supposed to depend on the quantum state. At first glance, the finiteness of the classical space seems paradoxical since the quantum state space is infinite. Indeed, the quantum information is not encoded into the classical state of a single execution but into the statistical behavior of many executions.

Like in a pilot-wave theory, in the presented model the system is supposed to be, at any instant, definitely in one of the two orthogonal states  $|\pm 1\rangle$ . More precisely, the model contains a two-value discrete index  $s = \pm 1$  that determines the outcome of a measurement on the basis  $\{|-1\rangle, |1\rangle\}$ . If at some time the index  $s$  is equal to some value  $s_0$  and a measurement is performed, then the outcome is  $|s_0\rangle$ . Furthermore, the measurement does not change the subsequent value of the index  $s$ . The role of  $s$  is similar to the role played by the position variables in the de Broglie–Bohm mechanics. To derive the stochastic model, we first introduce a simple measurement-noninvasive model that captures some features of a qubit. The qubit is described by just the bit  $s$  that is kicked by a time-correlated noise depending on the unitary evolution (first ingredient). Since this model is measurement-noninvasive, it satisfies the Leggett-Garg inequalities and it is not equivalent to quantum mechanics. We then modify the model by introducing another bit that is invasively updated by the measurements (second ingredient). The dynamics of  $s$  is ruled by both the noise and the additional bit. It is shown that this minimal improvement is sufficient for exactly reproducing the quantum transition between two consecutive measurements.

It is useful to represent the quantum states by Bloch vectors. The unitary evolution is described by a rotation on the Bloch sphere. We denote by  $R(t_a, t_b)$  the rotation operator along any time interval  $[t_a, t_b]$  and by vectors  $\pm \vec{n}$  the states  $|\pm 1\rangle$ . The quantum probability of having  $|s_1\rangle$  at time  $t_1$ , given  $|s_0\rangle$  at time  $t_0$ , is

$$P_Q(s_1, t_1 | s_0, t_0) = \frac{1}{2}[1 + s_1 s_0 \vec{n} \cdot \vec{v}], \quad (1)$$

$$\vec{v} \equiv R(t_0, t_1) \vec{n}. \quad (2)$$

Let us introduce the measurement-noninvasive model. First, we define the noise and the rule governing the jumps of  $s$ . Then, we derive the transition probability of  $s$  along a time interval. The noise variable is a unit vector  $\vec{x}(t)$  that is a function of time. We denote by  $\rho[\vec{x}(t)]$  the marginal probability distribution of  $\vec{x}(t)$  at time  $t$  and by  $\rho(\vec{x})$  the probability distribution of the function  $\vec{x}$  [i.e.,  $\rho(\vec{x})$  is a functional]. The statistical distribution of the noise, namely,  $\rho(\vec{x})$ , is defined by the equations

$$\rho[\vec{x}(t_a)] = (4\pi)^{-1}, \quad (3)$$

$$\vec{x}(t_b) = R(t_a, t_b) \vec{x}(t_a) \quad (4)$$

for any  $t_a$  and  $t_b$ . The first equation gives the marginal probability distribution of  $\vec{x}(t_a)$  at time  $t_a$ . The second

equation establishes a deterministic relation between the value of the noise variable at different times. The procedure for generating each realization of the noise is as follows. First, we generate a random vector  $\vec{x}(t_a)$  at some time  $t_a$  according to Eq. (3). Then, we determine  $\vec{x}(t)$  at any time by using Eq. (4). The noise is clearly time-correlated; that is, the correlation function  $\langle x_i(t_a) x_j(t_b) \rangle$  is not equal to zero for  $t_a \neq t_b$ . Given the noise function  $\vec{x}(t)$ , we need a rule for the dynamics of  $s(t)$ . We employ the simplest deterministic rule by assuming that the index  $s(t)$  undergoes a jump whenever  $\vec{n} \cdot \vec{x}(t)$  changes the sign, that is, whenever  $\vec{x}(t)$  crosses the geodesic of the Bloch sphere lying on a plane orthogonal to  $\vec{n}$ . Let us summarize the noninvasive model.

*Model 1.*—Let the unit vector  $\vec{x}(t)$  be a stochastic function of time, whose statistical property is given by Eqs. (3) and (4). In each Monte Carlo execution, the two-value index  $s(t)$  undergoes a jump whenever  $\vec{x}(t) \cdot \vec{n}$  changes the sign.

Thus, the Monte Carlo procedure for generating the value of  $s(t_1) = s_1$  at time  $t_1$  given  $s(t_0) = s_0$  at time  $t_0$  is as follows. A noise function  $\vec{x}(t)$  is generated according to Eqs. (3) and (4). If the signs of  $\vec{x}(t_0) \cdot \vec{n}$  and  $\vec{x}(t_1) \cdot \vec{n}$  are equal (even number of jumps), then  $s_1$  is set equal to  $s_0$ ; otherwise,  $s_1$  is set equal to  $-s_0$ . The procedure is repeated for each realization.

Notice that all we need to know about the noise is the real function  $\xi: t \rightarrow \vec{n} \cdot \vec{x}(t)$ . In fact, we could just suppose that  $\xi$  is the noise and regard  $\vec{x}$  as an intermediary tool for mapping each unitary evolution to a statistical distribution of  $\xi$ , namely, for generating the map  $R \rightarrow \rho(\xi)$  from the function  $R: t_a, t_b \rightarrow R(t_a, t_b)$  to the probability distribution (which is a functional) of the noise  $\xi$ . As with  $\vec{v}$  and  $\vec{v}(t)$ , if not differently indicated, we denote by  $\xi$  the noise function and by  $\xi(t)$  its value at time  $t$ . Thus,  $F(\xi)$  is meant as a functional of the noise at every time and  $F[\xi(t)]$  as a function of the noise at time  $t$ . Notice that the process  $\xi$  is not Markovian; that is, the marginal probability  $\rho(\xi_c | \xi_b, \xi_a)$  of having  $\xi_c = \xi(t_c)$  at time  $t_c$  given  $\xi_b = \xi(t_b)$  and  $\xi_a = \xi(t_a)$  at previous times  $t_b$  and  $t_a$  is not equal, in general, to the marginal probability  $\rho(\xi_c | \xi_b)$ .

Let us denote by  $P_\xi(s_1; t_1 | s_0; t_0)$  the transition probability from  $s_0$  at time  $t_0$  to  $s_1$  at time  $t_1$ , given a noise realization  $\xi$ . The rules defining model 1 imply that  $P_\xi(s_1; t_1 | s_0; t_0) \equiv \theta[s_1 s_0 \vec{n} \cdot \vec{x}(t_1) \vec{n} \cdot \vec{x}(t_0)]$ ,  $\theta$  being the Heaviside function. Using Eqs. (2) and (4), we have that

$$P_\xi(s_1; t_1 | s_0; t_0) = \theta[s_1 s_0 \vec{n} \cdot \vec{x}(t_1) \vec{v} \cdot \vec{x}(t_1)]. \quad (5)$$

From Eq. (5) and the statistical distribution of  $\vec{x}(t_1)$  defined by Eq. (3), we find, by the marginalization over  $\vec{x}(t_1)$ , that the probability of the transition  $s_0 \rightarrow s_1$  is

$$\begin{aligned} P(s_1, t_1 | s_0, t_0) &= \frac{1}{4\pi} \int d^2 x \theta[s_1 s_0 (\vec{n} \cdot \vec{x})(\vec{v} \cdot \vec{x})] \\ &= 1 - \frac{1}{\pi} \arccos(s_1 s_0 \vec{n} \cdot \vec{v}). \end{aligned} \quad (6)$$

Thus, the model does not exactly reproduce the quantum probability given by Eq. (1). For example, in the case of Rabi oscillation between states  $|\pm 1\rangle$ , the quantum probability is a cosine squared function of  $\omega(t_1 - t_0)$ ,  $\omega$  being the Rabi frequency. Conversely, the model presented here gives a triangle function. In particular, for a small evolution time the classical probability scales as  $t_1 - t_0$ , whereas the quantum probability scales as  $(t_1 - t_0)^2$ . While the model is not exact, it has the nice property of generating an oscillatory dynamics, which a Markov process on a finite set of states fails to give. This property is granted by the time correlation of the noise.

Just as a local model satisfies the Bell inequalities, this measurement-noninvasive model satisfies the Leggett-Garg inequalities, which are violated by quantum systems. To simulate exactly the quantum transition between two measurements we need to introduce some additional variable that is invasively updated by the first measurement. We now present an exact model that uses just one additional bit, denoted by a discrete index  $r(t)$ , taking the values  $\pm 1$ . The model of a qubit is as follows.

*Model 2.*—Let the unit vectors  $\vec{x}_1(t)$  and  $\vec{x}_{-1}(t)$  be two stochastic functions of time. They are statistically independent, and the statistical property of each function is given by Eqs. (3) and (4). The qubit is described by two indices  $s(t) = \pm 1$  and  $r(t) = \pm 1$ , which are functions of time  $t$ . The index  $s(t)$  is directly measurable at any time and is not modified by a measurement, whereas  $r(t)$  is invasively updated. If a measurement of  $s(t_0) \equiv s_0$  is performed at time  $t_0$ , then the index  $r(t_0)$  is set equal to

$$r_0 = \text{sgn}\{[\vec{x}_1(t_0) \cdot \vec{n}]^2 - [\vec{x}_{-1}(t_0) \cdot \vec{n}]^2\}. \quad (7)$$

The index  $r(t)$  remains constant after the measurement, whereas  $s(t)$  undergoes a jump whenever  $\vec{x}_{r_0}(t) \cdot \vec{n}$  changes the sign. A second measurement reveals the value of  $s(t_1) \equiv s_1$  at time  $t_1 > t_0$ .

Like in the previous model, all we need to know about the noise are the functions  $\xi_{\pm 1}: t \rightarrow \vec{n} \cdot \vec{x}_{\pm 1}(t)$ . In the following, we will denote by  $\xi$  the pair of noise functions  $\xi_{\pm 1}$ . A schematic representation of the model for a particular realization of the noise is given in Fig. 1. Notice in figure that a measurement at time  $t_0$  sets  $r = -1$ , since  $\xi_{-1}^2(t_0) > \xi_1^2(t_0)$ , in accordance with Eq. (7).

For each noise realization  $\xi$ , the probability of having outcome  $s_1$  at time  $t_1$ , given outcome  $s_0$  at time  $t_0$ , is equal to  $P_{\xi}(s_1; t_1 | s_0; t_0) = \theta[s_1 s_0 \vec{n} \cdot \vec{x}_{r_0}(t_1) \vec{n} \cdot \vec{x}_{r_0}(t_0)]$ , where  $r_0$  is given by Eq. (7). Thus, using Eqs. (2) and (4), we have that

$$P_{\xi}(s_1; t_1 | s_0; t_0) \equiv \theta[s_1 s_0 \vec{n} \cdot \vec{x}_{r_0}(t_1) \vec{v} \cdot \vec{x}_{r_0}(t_1)]. \quad (8)$$

Similarly, Eq. (7) can be written, through Eqs. (2) and (4), as

$$r_0 = \text{sgn}\{[\vec{x}_1(t_1) \cdot \vec{v}]^2 - [\vec{x}_{-1}(t_1) \cdot \vec{v}]^2\}. \quad (9)$$

Like in the previous model, the probability  $P(s_1; t_1 | s_0; t_0)$  for the transition from  $s_0$  at time  $t_0$  to  $s_1$  at

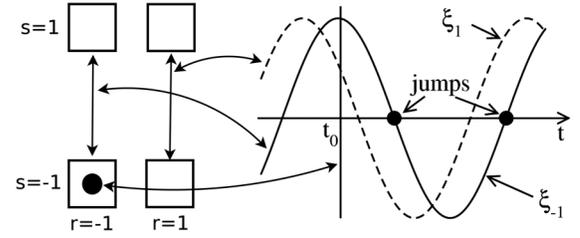


FIG. 1. Schematic representation of the 4-state model. During a unitary evolution the index  $s$  undergoes jumps whenever  $\xi_r$  changes sign, whereas  $r$  remains constant. For example, if a measurement is made at time  $t_0$  and  $r$  is set equal to  $-1$ , the jumps occur at the filled circles.

time  $t_1$  is obtained by averaging over the noise realizations. Thus, from Eqs. (3), (8), and (9) we have that

$$P(s_1; t_1 | s_0; t_0) = \frac{1}{(4\pi)^2} \sum_{r=\pm 1} \int d^2x_1 d^2x_{-1} \theta(s_1 s_0 \vec{n} \cdot \vec{x}_r \vec{v} \cdot \vec{x}_r) \times \theta\{r[(\vec{x}_1 \cdot \vec{v})^2 - (\vec{x}_{-1} \cdot \vec{v})^2]\}. \quad (10)$$

Noting that the two terms in the sum over  $r$  give the same contribution, it is not difficult to show that

$$P(s_1; t_1 | s_0; t_0) = \int d^2x \theta(s_1 s_0 \vec{n} \cdot \vec{x} \vec{v} \cdot \vec{x}) I(\vec{v} \cdot \vec{x}) \quad (11)$$

with  $I(\eta) \equiv \frac{1}{8\pi^2} \int d^2y \theta[\eta^2 - (\vec{v} \cdot \vec{y})^2] = \frac{1}{2\pi} |\eta|$ . This gives the equation

$$P(s_1; t_1 | s_0; t_0) = \int d^2x \theta(s_0 \vec{n} \cdot \vec{x}) \rho_{ks}(\vec{x}; s_1 \vec{v}), \quad (12)$$

where  $\rho_{ks}(\vec{x}; \vec{w}) \equiv \frac{1}{\pi} \vec{w} \cdot \vec{x} \theta(\vec{w} \cdot \vec{x})$  is the probability distribution associated with quantum state  $\vec{w}$  in the Kochen-Specker model [8]. The integral in Eq. (12) is well known [8] and gives the quantum probability of having the state  $s_0 \vec{n}$ , given the state  $s_1 \vec{v}$  and vice versa. Thus, we have proved that

$$P(s_1; t_1 | s_0; t_0) = P_Q(s_1; t_1 | s_0; t_0);$$

that is, the stochastic model exactly reproduces the quantum transition between two measurements. Unlike in a pilot-wave theory, the noise  $\xi$  does not depend on measurement times  $t_1$  and  $t_2$ . Thus, the noise value  $\xi(t)$  at any time  $t$  is independent of the quantum state at that time. Indeed the information on the quantum state is encoded in the correlation between  $(s, r)$  and  $\xi$ .

There is a close relation between this model and some results in quantum communication. Let us consider the following Einstein-Podolsky-Rosen scenario [6]. Two spatially separate parties, Alice and Bob, each receive one of two maximally entangled qubits. Alice performs a local projective measurement on the single-qubit basis  $(\vec{v}_1, -\vec{v}_1)$ , while Bob on the basis  $(\vec{v}_0, -\vec{v}_0)$ . The two-value indices  $s_0$  and  $s_1$  are defined so that Alice and Bob's outcoming states are the Bloch vectors  $-s_1 \vec{v}_1$  and

$s_0\vec{v}_0$ , respectively. With a suitable choice of the reference frame on the Bloch spheres, the joint probability distribution of  $s_0$  and  $s_1$  is  $P_e(s_0, s_1|\vec{v}_0, \vec{v}_1) = \frac{1}{4}[1 + s_0s_1\vec{v}_0 \cdot \vec{v}_1]$ . According to Bell's theorem, a local hidden variable model cannot reproduce this probability distribution, and some post-measurement communication between the parties has to be exchanged. How much communication is required? In Ref. [9], it was shown that a finite amount of communication, namely, 8 bits, is sufficient for reproducing the Bell correlations. This result was improved in Ref. [7], where it was shown that an exact simulation demands a communication of just 1 bit. An alternative model with minimal communication was derived in Ref. [10] from the Kochen-Specker model [8]. The common setting of a one-way classical protocol for simulating entanglement is as follows. Bob and Alice share a random variable  $X$ . Given the measurement  $(\vec{v}_0, -\vec{v}_0)$ , Bob generates the outcome  $s_0$  and an additional discrete index  $m$  according to a probability distribution that depends on  $\vec{v}_0$  and the shared variable  $X$ . Then he sends  $m$  to Alice. Alice generates the outcome  $s_1$  of the measurement  $(\vec{v}_1, -\vec{v}_1)$  with a probability that depends on both  $\vec{v}_1$  and  $X$  and the communicated index  $m$ . A stochastic model of quantum dynamics on a finite classical space, such as that introduced in this Letter, can be easily converted into a model of entanglement, where the stochastic noise and the classical state play the role of  $X$  and the communicated information, respectively.

Let us show that the stochastic model of qubit derived here can be converted into the model of entanglement reported in Ref. [10]. Suppose that at time  $t_0$  the qubit is in the mixture  $\frac{1}{2}(|1\rangle\langle 1| + |-1\rangle\langle -1|)$  and a projective measurement is performed on the basis  $(\vec{n}, -\vec{n})$ . The probability distribution of outcome  $s_0$  is  $\rho(s_0) = \frac{1}{2}$ . Then the qubit undergoes two consecutive unitary evolutions along the time intervals  $[t_0, t]$  and  $[t, t_1]$ . At time  $t_1$  another measurement on the basis  $(\vec{n}, -\vec{n})$  is made, and it gives outcome  $s_1$ . Each measurement at time  $t_i$ , with  $i = 1, 2$ , is actually equivalent to a measurement on the basis  $(\vec{v}_i, -\vec{v}_i)$  at the same time  $t$ , where  $\vec{v}_i \equiv R(t_i, t)\vec{n}$ . Thus, we have that the joint probability of  $s_0$  and  $s_1$  is formally equal to the probability distribution  $P_e(s_0, s_1|\vec{v}_0, \vec{v}_1)$  for two entangled qubits in the Einstein-Podolsky-Rosen scenario. Indeed, both the marginal distributions and the correlations of  $s_0$  and  $s_1$  are identically reproduced. From our stochastic model we find that, given the vectors  $\vec{x}_i(t) \equiv \vec{y}_i$ , the joint probability distribution of  $(s, r)$  at time  $t$  and outcome  $s_0$  at time  $t_0$  is

$$P_B(s_0, s, r|y; \vec{v}) = \frac{1}{2}\theta[(s_0\vec{v} \cdot \vec{y}_r)(s\vec{n} \cdot \vec{y}_r)]\theta\{r[(\vec{v} \cdot \vec{y}_1)^2 - (\vec{v} \cdot \vec{y}_{-1})^2]\}, \quad (13)$$

where  $y \equiv (\vec{y}_1, \vec{y}_{-1})$ . Similarly, we have that, given  $y$  and  $(s, r)$ , the probability of the outcome  $s_1$  at time  $t_1$  is

$$P_A(s_1|s, r, y; \vec{v}_1) = \theta[(s\vec{n} \cdot \vec{y}_r)(s_1\vec{v}_1 \cdot \vec{y}_r)]. \quad (14)$$

Finally, the joint probability of  $s_0$  and  $s_1$  is

$$P_e(s_0, s_1|\vec{v}_0, \vec{v}_1) = \sum_{s,r} \int d^4y P_A(s_1|s, r, y; \vec{v}_1) \times P_B(s_0, s, r|y, \vec{v}_0)\rho(y). \quad (15)$$

These three equations give a model of entanglement where  $y$  is the shared noise and  $(s, r)$  the communicated bits. Notice that  $\vec{n}$  is just a free parameter of the model. It can be eliminated by the transformation  $s \rightarrow s \operatorname{sgn}(\vec{n} \cdot \vec{y}_r)$ . Furthermore, the marginal probability distribution of  $s$  after the transformation is uniform and independent from  $\vec{v}$ . Thus,  $s$  can be included in the set of shared variables and, indeed, incorporated in  $y$ . In this way we obtain a local hidden variable model of entanglement, augmented by 1 bit of communication (namely,  $r$ ), given by the conditional probabilities

$$P_B(s_0, r|y; \vec{v}_0) = \theta(s_0\vec{v}_0 \cdot \vec{y}_r)\theta\{r[(\vec{v}_0 \cdot \vec{y}_1)^2 - (\vec{v}_0 \cdot \vec{y}_{-1})^2]\}, \\ P_A(s_1|r, y; \vec{v}_1) = \theta(s_1\vec{v}_1 \cdot \vec{y}_r).$$

This is the model derived in Ref. [10], set in a slightly different form. The process can be reverted, and one can obtain the stochastic model directly from the model of entanglement.

In conclusion, we have presented a stochastic model with time-correlated noise that exactly reproduces any unitary evolution of a qubit by using just 4 classical states. The time correlation of the noise allowed us to overcome the constraint of the theorem proved in Ref. [2] on the dimensionality of the classical space. A generalization to higher-dimensional quantum systems can have some interesting implications. First, it would automatically give a local hidden variable model of entanglement, augmented by a finite amount of one-way communication, for  $n$  Bell states. Apart from some approximate protocols reported in Ref. [10], such a model is at present missing. Second, this generalization can suggest more efficient methods for simulating the dynamics of high-dimensional quantum systems. At first glance, this approach does not seem to provide any computational benefit. Indeed, the evaluation of each noise realization requires one to solve the Schrödinger equation; thus, it is not less complicated than directly solving the dynamics of the quantum state. However, we have seen that only partial information about the noise (in our model, the real functions  $\xi_i$ ) is actually involved in the dynamics of the discrete indices. Thus, one could envisage a computational strategy for computing, exactly or with some approximation, this partial information without passing through the Schrödinger equation.

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