Non-Markovianity-Assisted Steady State Entanglement

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We analyze the steady state entanglement generated in a coherently coupled dimer system subject to dephasing noise as a function of the degree of Markovianity of the evolution. By keeping fixed the effective noise strength while varying the memory time of the environment, we demonstrate that non-Markovianity is an essential, quantifiable resource that may support the formation of steady state entanglement whereas purely Markovian dynamics governed by Lindblad master equations lead to separable steady states. This result illustrates possible mechanisms leading to long-lived entanglement in purely decohering, possibly local, environments. We present a feasible experimental demonstration of this noise assisted phenomenon using a system of trapped ions.

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The generation and ultimate persistence of quantum entanglement is normally thought to be correlated with a high degree of system isolation, while the presence of a surrounding environment tends to decohere the quantum system thus driving it towards a classically correlated state. However, driven systems, generally out of equilibrium, have been shown to tend towards steady states where quantum correlations are nonvanishing [1]. A variety of entanglement preserving mechanisms have been put forward, showing how the presence of environmental noise can be instrumental in keeping the system entangled in the steady state [1–7]. In the framework of a Markovian dynamics, the presence of local pure dephasing is normally detrimental for entanglement preservation [8]. The situation though is very different when the dephasing noise is non-Markovian. We show that the presence of steady state entanglement can be linked unambiguously to an increasing degree of non-Markovianity, even when the environments are acting locally. This result is of particular interest in the light of identifying mechanisms that assist entanglement preservation in condensed matter and biomolecular systems, where non-Markovian dephasing is a dominant noise source [9–13].

The system.—We consider a dimer system made up of two qubits (pseudo-spin-1/2 particles) coherently coupled via an exchange interaction of strength J, so that the system Hamiltonian reads $H_s = \sum_{j=1}^{2} \omega_j \sigma_j^+ \sigma_j^- + J(\sigma_1^- \sigma_2^+ + \sigma_1^+ \sigma_2^-)$ ($\hbar = 1$), where $\sigma_j^{\pm}$ are the raising (lowering) operators for site j, and subject to the action of an environment that leaves the populations unaffected but tends to randomize the phases of superposition states (the so-called pure dephasing or transverse decoherence). We will model this situation by subjecting each element $j = 1, 2$ of the dimer to the action of a *localized* harmonic mode, being the system-mode interaction governed by a Hamiltonian of the form $H_{s\text{-}m} = g_j \sigma_z^j (a_j + a_j^\dagger)$, where a_j ($a_j^\dagger$) denote the the operators of annhilitation (creation) of mode excitations. We will assume that the local modes are damped by a conventional Markovian bath so that the global time evolution of the dimer and the vibrational modes is described by a Markovian master equation with a Liouvillian part accounting for the damping of the localized modes at a certain rate κ. Note, however, that tracing out the local mode leads to a density matrix for the dimer system that is in general no reproducible from a purely Markovian evolution for the dimer alone. We will show that the interplay between the entangling exchange interaction and the local dephasing will result in an entanglement dynamics that depends crucially on whether or not the action of the environment can indeed be described by merely subjecting the dimer to Markovian dephasing. In the limit where the coherent coupling g is smaller than the decay κ we find that the effective dephasing rate γ_{eff}, which quantifies the strength of the system-environment coupling, is proportional to the ratio g^2/κ (see [14] for the case of cavity QED). When $2g \ll \kappa$ the dephasing is exponential and the decohering dynamics of the system can be reproduced by a Lindblad master equation with an effective decay rate γ_{eff}. For $2g < \kappa$, a regime which we denote as weakly non-Markovian, the definition of an effective dephasing rate γ_{eff} is still reasonable for times exceeding κ^{-1} when the decay is well approximated by an exponential. In the limit where $g > \kappa$ significant coherent oscillations can occur and the definition of an effective dephasing rate becomes meaningless. In this work we operate in the regime where $2g < \kappa$ and analyze the different dynamics that the dimer system undergoes when subject to the same effective noise strength but varying memory time of the environment. This will allow us to single out the precise influence of environmental time-correlations on the persistence/absence of stationary entanglement in the dimer system.

A convenient parametrization.—In order to link the observed dynamics to a parameter that quantifies how much

the environmental action departs from strict Markovianity, we will introduce an index f such that the coherent spin-mode coupling g and the mode damping rate κ are given by, respectively: $g = \sqrt{f} g_0$ and $\kappa = f\kappa_0$. In this way, the noise strength $g^2/\kappa = g_0^2/\kappa_0$ is kept fixed while varying f from values much larger than 1 to much smaller than 1 leads to, respectively, Markovian and non-Markovian dynamics with, crucially, the same effective local dephasing rate γ_{eff}. In other words, the *strength* of the dephasing noise is kept fixed, but the underlying noisy dynamics is modified, making the coherent system-oscillator coupling to dominate over losses (small f domain) or vice versa (limit of large f). Note, however, that while the decoherence rate is kept constant as the parameter f is varied, the population decay (which depends on g and κ) does change, as shown in Fig. 1. The monotonic change in Markovianity of the noise with the parameter f is shown in Fig. 2, where an explicit measure of non-Markovianity is considered.

Analytical results.—The effective Hamiltonian for the dimer-local damped modes system takes the form

$$\begin{aligned} H_{\text{eff}} &= \omega_1 \sigma_1^+ \sigma_1^- + \omega_2 \sigma_2^+ \sigma_2^- + J(\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \\ &\quad + (\Omega_1 - i\kappa_1) a_1^\dagger a_1 + (\Omega_2 - i\kappa_2) a_2^\dagger a_2 \\ &\quad + g_1 \sigma_1^{(z)} (a_1 + a_1^\dagger) + g_2 \sigma_2^{(z)} (a_2 + a_2^\dagger), \end{aligned} \tag{1}$$

where ω_i and Ω_i $(i = 1, 2)$ denote the site and the mode frequencies, respectively, and

$$\dot{\rho} = -iH_{\text{eff}}\rho + i\rho H_{\text{eff}}^\dagger + 2\kappa_1 a_1 \rho a_1^\dagger + 2\kappa_2 a_2 \rho a_2^\dagger. \tag{2}$$

Motivated by the situations frequently encountered in biomolecular complexes where external illumination is either weak and/or doubly excited states are strongly suppressed [15], we will focus here on situations for which the dimer dynamics is confined to the single excitation sector so that the entire dynamics for the two sites takes place in the subspace spanned by the states $|01\rangle$, $|10\rangle$. By introducing the convenient delocalized states $|u\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|d\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and the operators

$$\sigma_{p/m} = \frac{1}{\sqrt{2}}(\sigma_1^{(z)} \pm \sigma_2^{(z)}), \tag{3}$$

$$a_{p/m} = \frac{1}{\sqrt{2}}(a_1 \pm a_2), \tag{4}$$

noting that $\sigma_p|u\rangle = \sigma_p|d\rangle = 0$ while $\sigma_m|u\rangle = \sqrt{2}|d\rangle$ and $\sigma_m|d\rangle = \sqrt{2}|u\rangle$, in the subspace spanned by $|u\rangle$ and $|d\rangle$, the Hamiltonian part accounting for the system-mode interaction is given by

$$H_{s\text{-}m} = \frac{g_-}{2}\sigma_m(a_p + a_p^\dagger) + \frac{g_+}{2}\sigma_m(a_m + a_m^\dagger), \tag{5}$$

where $g_\pm = g_1 \pm g_2$. Note that the restriction to the one excitation sectors removes the contribution from terms involving σ_p, as this operator does not couple to the states $|u\rangle$ and $|d\rangle$. In the special case where $\omega_1 = \omega_2 = \omega$, $\Omega_1 = \Omega_2 = \Omega$, $\kappa_1 = \kappa_2 = \kappa$ and $g_1 = g_2 = g$, the total Hamiltonian takes the form

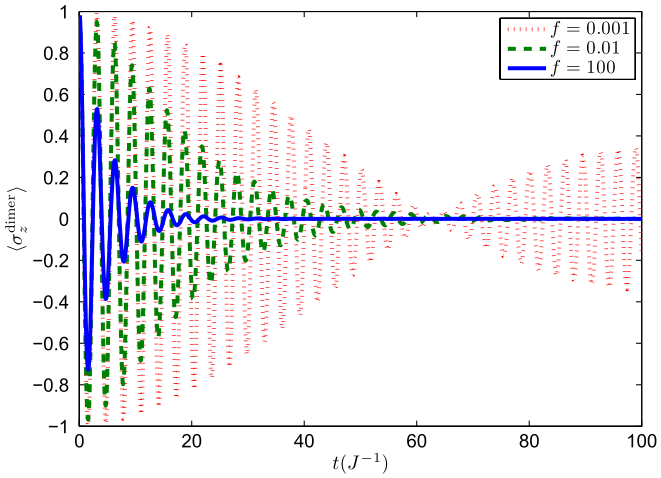

FIG. 1 (color online). Beating pattern in the dimer population inversion for different values of f (See main text for details). A Markovian environment ($f \gg 1$) will simply wash out the discreteness of the vibrational modes while the presence of beating in the dimer signal is a signature of the persistence of a coherent interaction with a localized vibrational (damped) mode.

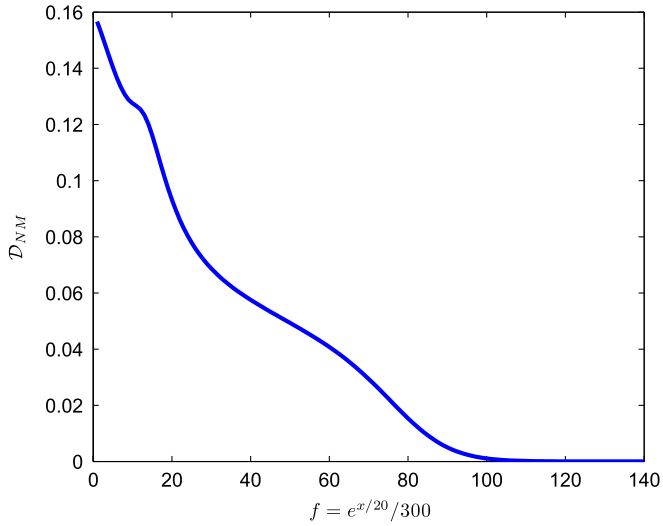

FIG. 2 (color online). Using the non-Markovianity measure introduced in [16] we can rigorously quantify the deviation from strict Markovianity of the dynamical map describing the time evolution of the dimer system. A finite value of the measure $\mathcal{D}_{NM}$ indicates that the dynamics cannot be accounted for in terms of a master equation of the Lindblad form. Here the values of f range from $f_{\text{min}} = 0.0035$ to $f_{\text{max}} = 3.6554$ when the coordinate x varies between 0 and 140. The value of $\mathcal{D}_{NM}$ decreases monotonically with f for the selected parameter regime. For f sufficiently large, the dynamical map is divisible, $\mathcal{D}_{NM}$ is zero and the dimer's evolution is fully Markovian.

$$H_{\text{eff}} = (\omega + J)|u\rangle\langle u| + (\omega - J)|d\rangle\langle d| + (\Omega - i\kappa)a_m^\dagger a_m + \sqrt{2}g(|u\rangle\langle d| + |d\rangle\langle u|)(a_m + a_m^\dagger), \quad (6)$$

and the time evolution reads,

$$\dot{\rho} = -iH_{\text{eff}}\rho + i\rho H_{\text{eff}}^\dagger + 2\kappa a_m \rho a_m^\dagger. \quad (7)$$

An analogous derivation could be done for the case where both subsystems in the dimer couple to a global mode. In this case, $H_{s\text{-}m} = \sigma_m g(a + a^\dagger)$, and therefore the dimer would remain decoupled if $g_1 = g_2$, while different couplings would yield to a system's dynamics analogous to that of the effective qubit ($|d\rangle$, $|u\rangle$) discussed above with $g = g_1 - g_2$. Figure 1 depicts the time evolution of the dimer's population inversion $\sigma_z^{\text{dimer}} = |10\rangle\langle 10| - |01\rangle\langle 01|$ for different values of f. For this example the following parameters were considered: $\omega = 0$, $J = 1$ (interdimer coupling), $\Omega = 2J$, $g_0 = J$ and $\kappa_0 = 20J$. With this, $\gamma_{\text{eff}} = J/10$. We truncated each oscillator after Fock layer 2 as the probability to have a single excitation in either mode never exceeds 10% in the parameter regime that we consider here. Varying f in the range from 10^{-3} to 100 allows us to move from a situation where $g \ll \kappa$ to the domain where $g \sim \kappa$, so that the damped mode ranges from being Markovian to imprinting an element of non-Markovianity to the dynamics, as quantitatively exemplified when evaluating the degree of non-Markovianity $\mathcal{D}_{NM}$ introduced in [16]. This measure provides a necessary and sufficient condition for a given evolution to depart from strict Markovianity by means of evaluating whether or not the associated dynamical map $\mathcal{E}_{(t+\epsilon,t)}$ is completely positive (CP) for any ϵ. For that, it has to be $(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathbb{1})|\Phi\rangle \times \langle\Phi| \geq 0$, where $|\Phi\rangle$ is maximally entangled of our open system and some ancillary system [17]. Using this condition, one can define a quantitative measure to quantify how much does the evolution departs from strict Markovianity by evaluating $\mathcal{I} = \int_0^\infty g(t)dt$, where the function g quantifies how much does the norm of state $(\mathcal{E}_{(t+\epsilon,t)} \otimes \mathbb{1})|\Phi\rangle\langle\Phi|$ differ from 1 in the limit of $\epsilon \to 0$ [16]. In Fig. 2, we have plotted the normalized measure $\mathcal{D}_{NM} = \frac{\mathcal{I}}{\mathcal{I}+1}$ for a range of values of f, showing that the domain of large f leads to the dimer's dynamics to be fully Markovian. Limiting the harmonic oscillator to just 2 levels, we find for the steady state population of the state $|d\rangle$ the result

$$\rho_{dd}^{ss} \sim \frac{4g^2 + \kappa^2 + (2J + \Omega)^2}{2(4g^2 + \kappa^2 + 4J^2 + \Omega^2)}. \quad (8)$$

In the limit of sufficiently small f, and in the regime where the local mode is quasiresonant with the dimer eigenstates, so that $\Omega \sim J$, one finds that the above expression tends to unity. Hence, in this picture, the system is in weak interaction with a composite environment (local mode + Markovian reservoir) and relaxes towards its ground state, which in this case is the singlet state. When weakly coupled to a purely Markovian environment and experiencing dephasing at the same rate as before, the steady state of the dimer system tends towards the maximally mixed state and no entanglement survives in the long term. The time evolution of the entanglement in the dimer system, as quantified by the logarithmic negativity [18] of the state, is summarized in Fig. 3, for an initial factorized condition where there is an excitation in one site only. Indeed, when f is large, the dynamics of the dimer can be reproduced by simulating a Markovian master equation with local dephasing rates γ_{eff}. In this case, the steady state of the dimer is maximally mixed, a result that can be proven using the results presented in section 4 of [8], following theorem 5.2 of [19]. When f decreases but remaining in the weakly non-Markovian regime, the steady state approaches the singlet state, which in this case is the lowest energy eigenstate of the coherent evolution. Provided that we keep on the weak non-Markovian regime (so that the condition $\kappa_i > g_i$ is satisfied), altering the symmetry of the problem and considering $g_1 \neq g_2$ does not reduce the overlap with the singlet below 90% up to ratios $g_2/g_1 = 2$. At a finite temperature, the steady state entanglement is decreased but remains finite. For an average photon number of $n_{\text{th}} = 0.3$ we find that the steady state entanglement reaches the value of 0.67 for $f = 0.01$ (Finite T results are not shown in the figures). For GHz frequencies as they are typical for quantum optical implementations this corresponds to around 300 mK. For typical biological systems however, the bath spectral density peaks around 200 cm^{-1} (corresponding to $\omega = 4 \times 10^{13}$ Hz) so that $n_{\text{th}} = 0.3$ corresponds to a temperature $T \cong 77$ K. At these temperatures typical biomolecular systems such as

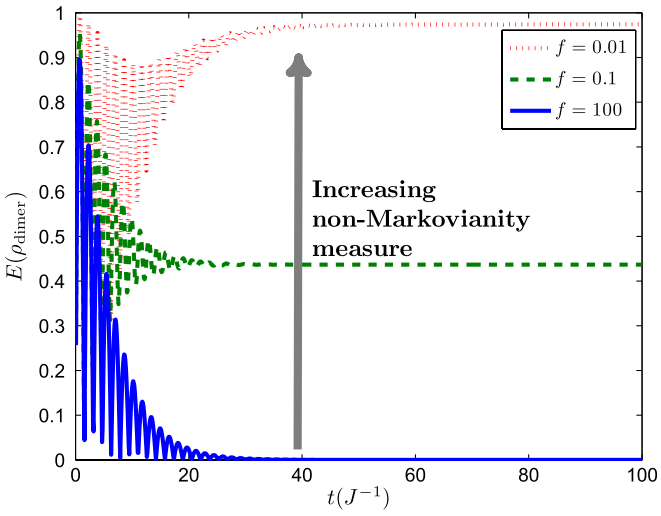

FIG. 3 (color online). Time evolution of the entanglement content of the dimer system when the index f varies from the range $f \gg 1$ (Markovian evolution) to $f \ll 1$ (non-Markovian effects). When the evolution is strictly Markovian, the steady state is separable while in the presence of non-Markovianity, the dimer system contains quantum correlations that steadily become close to 1 e bit for f sufficiently small.

the Fenna-Matthew-Olson complex will suffer non-Markovian system-environment dynamics and exhibits long-lived coherences in the dynamics as demonstrated in recent experiments [20].

Possible experimental realization.—In this section we outline a possible implementation of the model dynamics discussed in the previous section. This realization employs proven elements of ion trap technology and offers, in principle, the possibility for control of all the relevant parameters in the Hamiltonian Eq. (2). Three basic dynamical elements have to be generated. The electronic degrees of freedom of trapped ions constitute the dimer and need to be coupled to each other, the motional degrees of freedom play the role of the environment and need to be coupled to the electronic degrees of freedom via a dispersive interaction and laser cooling needs to be employed to induce damping of the motional degrees of freedom which generates a Lorentzian spectral density.

For the implementation of the direct coupling between the two qubits we make use of a Sørensen-Mølmer gate [21,22]. To this end one applies laser fields with two different frequencies so that the two-photon process coupling $|gg\rangle \leftrightarrow |ee\rangle$ and $|eg\rangle \leftrightarrow |ge\rangle$ is resonant, i.e., $\omega_1^L + \omega_2^L = 2\omega_{eg}$, while neither of the frequencies are resonant with single excitations of the ions. This can be achieved by choosing $\omega_1^L = \omega_{eg} - \delta$ and $\omega_2^L = \omega_{eg} + \delta$, where $\delta \ll \eta\Omega_R$, ν. Here η denotes the Lamb-Dicke parameter, Ω_R the laser Rabi frequency and ν is the frequency of the center-of-mass mode. If we prepare the two ions representing the dimer initially in a single excitation subspace we realize in this fashion the Hamiltonian $H_{\text{dimer}} = J_{\text{eff}}(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-)$ with an effective exchange coupling $J_{\text{eff}} \sim (\Omega_R\eta)^2/(\nu - \delta)$ [21]. The added advantage of this scheme is the fact that it does not require the center-of-mass mode to be in its ground state thus increasing the robustness of the scheme. The excitation preserving coupling Hamiltonian $\sigma_z(a + a^\dagger)$ between an ion and the motional degrees of freedom can be achieved in several ways. One approach [23] subjects the ion to a far off-resonant standing wave, which creates the state dependent potential $V(z) = V_0\cos^2(k\hat{z} + \frac{\pi}{4})\sigma_z$. Here k is the wave-vector of the standing wave lasers. The operator $\hat{z}$ is readily expressed in terms of phonon operators, $z = \sum_n \mathcal{M}_n\sqrt{\frac{1}{2m\omega_n}}(a_n + a_n^\dagger)$, where m is the ion mass, and $\mathcal{M}_n$ is the amplitude of each vibrational mode n at the ion. Expanding this potential in the small parameter $kz \ll 1$ we find that the leading order contribution is linear in z with the second order contribution canceling. Hence we obtain a Hamiltonian $H_{sb} = \sigma_z\sum_n \mathcal{M}_n\sqrt{\frac{1}{2m\omega_n}}(a_n + a_n^\dagger)$. Traveling wave fields can also be used to generate this type of interaction in a suitably transformed basis [23]. This scheme couples to all the modes of the ion crystal which may lead to correlations between the environments acting on separate parts of the dimer. We may select specific environment modes by choosing light fields that are slightly off-resonant to specific modes while being more strongly detuned from the remaining modes. Finally our dimer model assumes that the modes coupling to the constituents of the dimer are damped. Damping of motional degrees of freedom can be achieved, of course, by applying laser cooling to an auxiliary third ion that couples to all the motional modes of the system. To be effective, this auxiliary ion should be placed at the end of the ion chain to ensure coupling to all modes in the ion string. Combining these three elements yields the dynamical Hamiltonian discussed in Eq. (2) which underlies the non-Markovianity driven steady state entanglement in a dimer system.

Conclusions.—We have demonstrated that, when subject to system-environment interactions of the same effective strength, the non-Markovian character of the noise can be the crucial property that leads to steady state entanglement where purely Markovian noise would result in the complete destruction of entanglement. A possible experimental verification of this dephasing-assisted phenomenon in ion trap physics which employs only experimentally demonstrated building blocks has been discussed. We expect these studies to contribute towards the identification of the physical mechanisms that could underpin the persistence of stationary quantum correlations in very noisy environments occurring in natural conditions [9–13]. A key issue in this context would be the evaluation of this effect for realistic (possibly strongly non-Markovian), multicomponent, biological systems operating at physiological temperatures. Numerical methods with the ability to simulate exactly structured spectral densities at finite temperatures would be required to fully address this issue and initial steps towards this development have already been presented [24].

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