Rapidity Renormalization Group

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We introduce a systematic approach for the resummation of perturbative series which involves large logarithms not only due to large invariant mass ratios but large rapidities as well. A series of this form can appear in a variety of gauge theory observables. The formalism is utilized to calculate the jet broadening event shape in a systematic fashion to next-to-leading logarithmic order. An operator definition of the factorized cross section as well as a closed form of the next-to-leading-log cross section are presented. The result agrees with the data to within errors.

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Observables in weakly coupled gauge theories often necessitate perturbative resummations to be under calculational control. This need arises when one performs measurements that are sensitive to infrared scales. By probing distances long compared to the hard scattering scale, one introduces large logarithms (logs) that lead to the breakdown of fixed-order perturbative series. Resumming the large logs has become standard in QCD [1] and can be accomplished by factorizing the cross section into momentum regions. Factorization makes clear the distinction between logs of various ratios that may be involved in the observable, and resummation follows via standard renormalization group techniques.

An elegant formalism for factorization is soft-collinear effective theory (SCET) [2], which is an effective field theory designed to reproduce the infrared physics in high-energy processes. The formalism not only streamlines factorization proofs [3] but also allows one to systematically include power corrections. The results of this Letter will all be couched in terms of this framework.

A generic factorized cross section takes on the form

$$\sigma = H \otimes [\Pi_i J_i] \otimes S. \tag{1}$$

The hard function H is responsible for reproducing the short-distance physics with wavelengths of the order of $1/Q$, where Q is the scale involved in the hard scattering. J_i and S are the so-called jet and soft functions containing modes which are highly energetic (collinear) and soft, respectively. Soft modes have small rapidities ($k_+/k_- \sim 1$), while the rapidities of collinear modes are parametrically larger ($k_\pm/k_\mp \gg 1$), where $k_\pm$ are the light-cone momenta. The tensor product implies the existence of one or more convolutions in momentum space. In canonical situations, the resummation of large logs is accomplished by evolving, via the renormalization group, each factorized component to its natural scale. The natural scales are set by the arguments of the logs. H, J, and S may contain, for example, logs of Q/μ, m_J/μ, and m_S/μ, respectively, where m_J and m_S are quantities which probe the invariant masses of the modes composing J and S.

While there is a large disparity in rapidity between the modes which compose S and J, the typical invariant mass of the modes needs not be distinct. When soft fields have invariant mass parametrically smaller than the collinear modes (in this case, the soft modes are called ''ultrasoft''), significant simplifications arise. Whether or not there is a distinction in invariant masses, one must always ensure that there is no double counting of modes. That is, loop integrals within a prescribed function (J or S) should only account for the relevant mode. In principle, this could be accomplished using a cutoff, but this would lead to multiple technical difficulties, not the least of which is the need for gauge noninvariant counterterms. Within the effective field theory formalism using dimensional regularization, this double counting is avoided by the so-called zero-bin procedure [4]. In this methodology, one subtracts from each loop integral its value when the integrand is asymptotically expanded around the extraneous region. This procedure not only formally avoids the double counting but also ensures that all integrals in the theory are well defined. Moreover, the zero-bin subtraction, or some equivalent subtraction method, is necessary to preserve factorization [5]. This potential breakdown of factorization occurs as a consequence of the need to regulate ''rapidity divergences'' (light-cone singularities). These divergences arise schematically from integrals of the form

$$I_R = \int \frac{dk_+}{k_+}, \tag{2}$$

which are not regulated in dimensional regularization. There are multiple ways of regulating this divergence. A simple way is to introduce a new dimensionful parameter Δ via the replacement $k_+ \to k_+ + \Delta$ in the denominator. Note that regulating these divergences will break boost invariance along the light-cone direction. For any physical observable, the final result must be boost invariant and independent of Δ; this is automatic once zero-bin subtraction is performed and all sectors added. For the case of $m_J \gg m_s$, Δ dependence cancels in each sector after zero-bin subtraction and the boost symmetry is restored in each sector. This will, however, not be the case when $m_J \sim m_S$, since the soft and collinear modes mix under boosts.

When $m_J \sim m_S$, the jet *axis*, defined as the direction of the net momentum of the jet, recoils against the soft emission. In the light-cone coordinates, collinear modes scale like $(n \cdot p, \bar{n} \cdot p, p_\perp) \sim Q(1, \lambda^2, \lambda)$, while soft modes scale as $Q(\lambda, \lambda, \lambda)$, where $\lambda \ll 1$ and n^μ is the light-cone direction chosen to perform the factorization. The jet axis is no longer aligned with n^μ, and one should not expect the jet function to be invariant under boosts along n^μ. However, the sum of all sectors will still be invariant. If we were to reanalyze the rapidity divergences discussed previously, we would find that the Δ regulator will cancel after summing over contributions from all sectors (with proper zero-bin subtractions) [5]. Boost noninvariance in the jet function appears in the form $\log[Q/\Delta]$, a "rapidity log." Resumming these logs using a Δ regulator is technically cumbersome, much like resummations with a cutoff regulator. Here, we introduce a regulator more in the spirit of dimensional regularization that does not introduce new dimensionful scales in the integrals and maintains manifest power counting in the effective theory.

Given the existence of the rapidity logs in addition to the canonical logs, S and J may not have one definitive scale associated with them. To resum both sets of logs, we will introduce another arbitrary scale ν, along the lines of μ in dimensional regularization. We expect that, in order to properly resum all the large logs, we will need to run the jet in ν down to the smaller rapidity scale of the soft function. In a Wilsonian sense, we have two distinct cutoffs with which to thin degrees of freedom. There will be one flow in invariant mass and one in rapidity, as shown in Fig. 1. This is an inherently Minkowskian procedure.

To illustrate this rapidity renormalization group, we will consider the specific example of the event shape called jet broadening. Event shapes have played an important role in precision measurements of the strong coupling α_s [6]. A generalized event shape for event $e^- e^+ \to X$ at center-of-mass energy $\sqrt{s}$ can be defined [7] in terms of a parameter a via

$$e(a) = \sum_{i \in X} \frac{|\vec{p}_{i\perp}|}{\sqrt{s}} e^{-|\eta_i|(1-a)}, \tag{3}$$

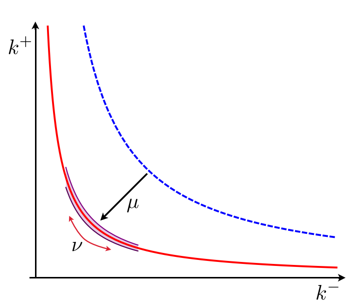

FIG. 1 (color online). Rapidity renormalization group flow along the on-shell hyperbola versus the standard flow.

where $p_{i\perp}$ is the transverse momentum with respect to the thrust axis of the event and η_i is the rapidity of the ith particle. The thrust axis $\hat{t}$ is defined via $\max_{\hat{t}} \sum_{i \in X} |p_i \cdot \hat{t}| / \sqrt{s}$. Two particularly interesting event shapes are the limits $a = 0, 1$ corresponding to "thrust" and "jet broadening," respectively. The limit $e \ll 1$ isolates events composed of back-to-back jets. In the case of thrust, these jets are composed of collinear radiation, and the recoil due to soft (ultrasoft, in this case) radiation does not affect the jet axis, while, for jet broadening, all radiation with parametrically similar transverse momentum can contribute, so that soft radiation of the form $Q(\lambda, \lambda, \lambda)$ recoils the jet off the thrust axis. In both of these cases, fixed-order perturbation theory will fail when e is small. However, as long as $eQ \gg \Lambda_{\mathrm{QCD}}$, we expect nonperturbative effects to be suppressed, although large logs of e need to be resummed.

The pioneering work on jet broadening resummations [8] utilized the coherent branching formalism [9]. It was later stated [10] that the results in [8] neglected terms due to the recoil of soft gluons. In this Letter, we will provide a factorization theorem for jet broadening, whose proof will follow in a subsequent publication [11]. The factorization proofs for angularity observables (3) in [12] are known to fail as a approaches 1, since there are growing power corrections in this limit where one approaches jet broadening. The reason for the apparent breakdown of factorization is the fact that, in this limit, the soft radiation has the same invariant mass as collinear radiation, and one must change the power counting accordingly to factorize in a consistent fashion [11].

Henceforth, we set $a = 1$ and $e(1) \equiv e$. In [11], we prove that the cross section for jet broadening takes the following factorized form:

$$\begin{aligned} \frac{d\sigma}{de} &= \sigma_0 H(s) \int de_n de_{\bar{n}} de_s \delta(e - e_n - e_{\bar{n}} - e_s) \\ &\quad \times \int dp_{1t} dp_{2t} J_n(Q_+, e_n, p_{1t}) J_{\bar{n}}(Q_-, e_{\bar{n}}, p_{2t}) \\ &\quad \times \mathcal{S}(e_s, p_{1t}, p_{2t}), \end{aligned} \tag{4}$$

where, in covariant gauges,

$$\begin{aligned} J_n &= \Omega_{\bar{d}} \int \frac{dx_+}{2N_c} e^{iQ^- x^+/2} \langle 0 | \bar{\chi}_n(x_+) \frac{\bar{\not{n}}}{2} \delta(\hat{e} - e_n) \\ &\quad \times \delta(\hat{P}_\perp + \vec{p}_{1\perp}) \chi_n(0) | 0 \rangle, \\ \mathcal{S} &= p_{1t}^{1-2\epsilon} p_{2t}^{1-2\epsilon} \Omega_{\bar{d}} \int \frac{d\Omega_{12}}{N_c} \langle 0 | S_n^\dagger S_{\bar{n}} \delta(\hat{e} - e_s) \\ &\quad \times \delta^{\bar{d}}(\hat{P}_{n\perp} - \vec{p}_{1\perp}) \delta^{\bar{d}}(\hat{P}_{\bar{n}\perp} - \vec{p}_{2\perp}) S_{\bar{n}}^\dagger S_n | 0 \rangle, \end{aligned} \tag{5}$$

and σ_0 is the Born cross section. H is the hard matching coefficient which incorporates all the short-distance contributions and is fixed by matching the QCD currents onto the SCET currents. Here, $\bar{d} = 2 - 2\epsilon$ and $\chi_{(n,\bar{n})}$ are gauge invariant SCET fields which include collinear Wilson lines $W_{n,\bar{n}}$ and the $S_{n,\bar{n}}$ are soft Wilson lines. $\hat{P}_{n\perp}$ and $\hat{P}_{\bar{n}\perp}$ are hemisphere-transverse momentum operators, $\Omega_{\bar{d}}$ refers to the area of a $\bar{d}$-dimensional unit sphere, Ω_{12} refers to the relative angle between $\vec{p}_{1\perp}$ and $\vec{p}_{2\perp}$, and $p_{it} = |\vec{p}_{i\perp}|$. Finally, $Q^{\pm}$ are the large light-cone momenta of the jets with constraint $Q^+Q^- = s$, the center-of-mass energy.

As long as $\sqrt{s}e \gg \Lambda_{\rm QCD}$, all of these matrix elements are calculable in perturbation theory. The bare matrix elements possess both rapidity and UV divergences. Thus, standard dimensional regularization is insufficient to regulate all the integrals. Beyond tree level, one is immediately met with the aforementioned rapidity divergences. To regulate these integrals, we introduce a regulator into the momentum space (Abelian) SCET Wilson lines as follows:

$$\begin{aligned} W_n &= \bigg[\sum_{\rm perm} \exp\bigg(\frac{-g}{\bar{n}\cdot\hat{P}}\bigg[w\frac{|\bar{n}\cdot\hat{P}|^{-\eta}}{\nu^{-\eta}}\bar{n}\cdot A_{n,q}(0)\bigg]\bigg)\bigg], \\ S_n &= \bigg[\sum_{\rm perm} \exp\bigg(\frac{-g}{n\cdot\hat{P}}\bigg[w\frac{|2\hat{P}^3|^{-\eta/2}}{\nu^{-\eta/2}}n\cdot A_{s,q}(0)\bigg]\bigg)\bigg], \end{aligned} \tag{6}$$

which suffices for one-loop calculations. Here, ν is an arbitrary scale independent of the usual scale μ introduced in dimensional regularization. w is a bookkeeping parameter which will be set to one at the end of the calculation. $\hat{P}^\mu$ is the momentum operator, and we have essentially regulated the longitudinal momenta of the emitted gluons in each Wilson line, since $|2P_3| \to \bar{n}\cdot P$ in the collinear limit. Notice the factor of $\eta/2$ in the soft function. This choice is not a matter of convention. Physically, the factor arises as a consequence of the fact that soft must be cut off at both positive as well as negative rapidity. For the non-Abelian case, we modify the regulator by regulating the longitudinal group momenta of maximally non-Abelian webs [11]. The Wilson line regularization breaks manifest boost symmetry, which is restored once all of the sectors are combined. The rapidity divergences for the jet and soft functions will introduce a new set of anomalous dimensions $(\gamma_J^\nu, \gamma_S^\nu)$ which are defined via variation of ν. Given that the hard function has no such anomalous dimensions, we must have the relation

$$2\gamma_J^\nu + \gamma_S^\nu = 0, \tag{7}$$

or, equivalently, it must be true that the total η dependence must vanish. Indeed, it is not hard to show [11] that the sum of the η divergences cancels as a consequence of eikonalization. This cancellation also implies that the individual factors J and S are multiplicatively renormalizable. Note that, while gauge invariance is not manifest in the regulated Wilson line, it can be shown [11] that the gauge-dependent pieces contain no rapidity divergences.

The tree-level jet function is given by $\delta(e_n - p_{1t}/\sqrt{s})$, and the soft function is given by $\delta(e_s)\delta(p_{1t})\delta(p_{2t})$. To determine the relevant scales in the logs, we must convolve the renormalized one-loop jet function with the tree-level soft function as dictated in (4). This is most easily seen for the integrated cross section $\Sigma = \int_0^{e_0} de(d\sigma/de)$. At the order of α_s, the result of the convolution leads to the following singular contributions from a jet:

$$\frac{1}{\sigma_0}\Sigma_{\rm jet} = \frac{-\alpha_s C_F}{2\pi}\ln\bigg(\frac{\sqrt{s}e_0}{2\mu}\bigg)\bigg[3 + 4\ln\bigg(\frac{\nu}{Q^{\pm}}\bigg)\bigg]. \tag{8}$$

It thus becomes clear that the jet function depends on two different kinematical scales ($\sqrt{s}e_0$ and $Q^{\pm}$). In addition, we see that dependence on rapidity manifests in the form of the rapidity log, $\ln(Q^{\pm}/\nu)$. The soft function singular contributions are

$$\frac{1}{\sigma_0}\Sigma_{\rm soft} = \frac{\alpha_s C_F}{\pi}\bigg[-2\ln^2\bigg(\frac{\sqrt{s}e_0}{2\mu}\bigg) + 4\ln\frac{\nu}{\mu}\ln\bigg(\frac{\sqrt{s}e_0}{2\mu}\bigg)\bigg] \tag{9}$$

and have rapidity logs set at the low scale $\sqrt{s}e_0$.

The utility of ν is clear, as we may choose $\nu \sim Q^{\pm} \sim \sqrt{s}$ and $\mu \sim \sqrt{s}e$ to minimize the logs in the jet function. Then, to minimize the logs in the soft function, we run ν from the scale $\sqrt{s}e$ up to $\sqrt{s}$. Furthermore, we will need to run the hard matching coefficient down to the scale $\sqrt{s}e$ in the dimensional regularization parameter μ. This scenario is shown schematically in Fig. 2.

Here, we will perform the running at next-to-leading log (NLL), which sums all terms of order one, where we take the scaling $\alpha_s\ln(e) \sim 1$. The next-to-next-to-leading-log (NNLL) analysis will be performed in [11]. The hard function renormalization group equation is well known to NNLL (see [13]). The running is most simply performed in Laplace transform space, with (b, b') conjugate to (p_{1t}, p_{2t}), respectively. We find at one loop

$$\gamma_\nu^S(b, b') = -2\Gamma_c^{(0)}[(\log(b\mu e^{\gamma_E}) + \log(b'\mu e^{\gamma_E})], \tag{10}$$

where $\Gamma_c^{(0)} = \alpha_s C_F/\pi$ is the cusp anomalous dimension for Wilson lines. Similar equations can be written for the two jet functions in terms of their corresponding anomalous dimensions.

In our strategy of resumming logs for the soft function, we only need a two-loop cusp. The solution of the ν renormalization group equation for the soft function is

$$\mathcal{S}(\mu, \nu) = V_s(\mu, \nu/\nu_0) \otimes \mathcal{S}(\mu, \nu_0), \tag{11}$$

where $\otimes$ represents convolution in kinematical arguments which are dropped for brevity. Here,

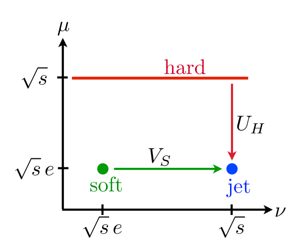

FIG. 2 (color online). Running strategy.

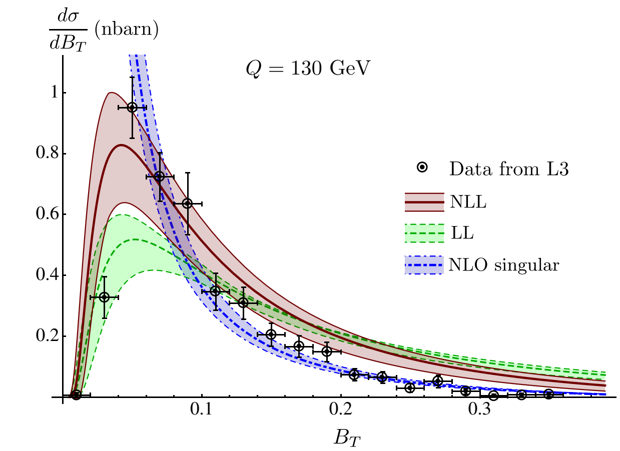

FIG. 3 (color online). Total jet broadening at 130 GeV.

$$\begin{aligned} &V_s(p_{1t}, p_{2t}; \omega_s(\mu, \nu/\nu_0)) \\ &= \frac{e^{-2\gamma_E\omega_s}}{\Gamma^2(1+\omega_s)}\left(\frac{\omega_s}{\mu}\left[\frac{1}{(\frac{p_{1t}}{\mu})^{1-\omega_s}}\right]_+ + \delta(p_{1t})\right) \\ &\quad\times\left(\frac{\omega_s}{\mu}\left[\frac{1}{(\frac{p_{2t}}{\mu})^{1-\omega_s}}\right]_+ + \delta(p_{2t})\right), \end{aligned} \tag{12}$$

with $\omega_s(\mu, \nu/\nu_0) = 2\Gamma_c[\alpha_s(\mu)]\log\frac{\nu}{\nu_0}$. To minimize logs in the hard function, we need to evolve the hard function using

$$H(s;\mu) = H(s;\mu_0)U_H(s;\mu_0,\mu), \tag{13}$$

where, up to NLL, U_H can be found in [13] and $H(s,\mu_0) = 1$ to the order we are working.

The results we have presented so far are for the angularity at $a = 1$, which is related to the total jet broadening B_T via $e = 2B_T$. We will present cross sections for total jet broadening here and compare with the data. For the next-to-leading-order singular cross section, we get

$$\frac{d\sigma}{dB_T} = \sigma_0\frac{\alpha_s(\mu)C_F}{\pi B_T}(-3 - 4\log B_T), \tag{14}$$

where σ_0 is the Born cross section. This result is in agreement with Ref. [8]. For the resummed cross section, up to NLL order, we have

$$\frac{d\sigma}{dB_T} = \frac{\sigma_0}{B_T}\frac{U_H(Q^2,\mu_Q,\mu)}{\Gamma(2\omega_s)e^{2\gamma_E\omega_s}}\left(\frac{QB_T}{\mu}\right)^{2\omega_s}. \tag{15}$$

The result differs from [10] only because we Laplace transformed in b and did not Fourier transform in $\vec{b}$, which would yield [11] results identical to those in [10]. However, note that the difference between the results in [8,10] is at most 10% (at the peak). Our results have the advantage that the scale dependence on μ and ν is manifest, allowing us to define a systematic theoretical error, which was absent in [10].

In Fig. 3, we have plotted the theory cross section and the data [14]. We see that, given the large error bars, the agreement with the data is reasonable. However, the NNLL calculation will reduce the theory errors considerably. We have not included the theory errors due to power corrections. In the small B_T region, these are nonperturbative and scale as $\Lambda_{\rm QCD}/(B_TQ)$ and can be expected to be of the order of 20–30%. In the tail region, there are corrections of the order of B_T relative to the singular contributions. The disagreement at intermediate values of B_T, where fixed-order calculations suffice, is expected, since logs will not dominate in this region and NLL results leave off order-one contributions. This region will be correctly reproduced in the NNLL calculation. Therefore, by systematically improving this result by including higher-order corrections in α_s, power corrections, and nonperturbative correction, this result can be used for precision α_s determination. Such an analysis using thrust was done in [13].

Finally, we wish to point out that the rapidity renormalization group can be utilized in multiple other settings where rapidity divergences arise. Generically, this will occur whenever kinematically soft radiation has invariant mass of the same order as the collinear radiation, as in cases where one measures the p_T of the final state. Such observables will be discussed in more detail in [11]. Furthermore, it would be interesting to utilize our rapidity renormalization group in the context of exclusive processes, where it has been shown that rapidity factorization sheds light on end point singularities in integrals over light-cone distribution functions [4].

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