Is a System's Wave Function in One-to-One Correspondence with Its Elements of Reality?

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Although quantum mechanics is one of our most successful physical theories, there has been a long-standing debate about the interpretation of the wave function—the central object of the theory. Two prominent views are that (i) it corresponds to an element of reality, i.e., an objective attribute that exists before measurement, and (ii) it is a subjective state of knowledge about some underlying reality. A recent result [M. F. Pusey, J. Barrett, and T. Rudolph, arXiv:1111.3328] has placed the subjective interpretation into doubt, showing that it would contradict certain physically plausible assumptions, in particular, that multiple systems can be prepared such that their elements of reality are uncorrelated. Here we show, based only on the assumption that measurement settings can be chosen freely, that a system's wave function is in one-to-one correspondence with its elements of reality. This also eliminates the possibility that it can be interpreted subjectively.

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Introduction.—Given the wave function associated with a physical system, quantum theory allows us to compute predictions for the outcomes of any measurement. Since a wave function corresponds to an extremal state and is therefore maximally informative, one possible view is that it can be considered an *element of reality* of the system, i.e., an objective attribute that exists before measurement. However, an alternative view, often motivated by the probabilistic nature of quantum predictions, is that the wave function represents incomplete (subjective) knowledge about some underlying reality. Which view one adopts affects how one thinks about the theory at a fundamental level.

To illuminate the difference between the above views, we give an illustrative example. Consider a meteorologist who gives a prediction about tomorrow's weather (for example, that it will be sunny with probability 33% and cloudy with probability 67%; see the left-hand side of Fig. 1). We may assume that classical mechanics accurately describes the relevant processes, so that the weather depends deterministically on the initial conditions. The fact that the prediction is probabilistic then solely reflects a lack of knowledge on the part of the meteorologist on these conditions. In particular, the forecast is not an element of reality associated with the atmosphere but rather reflects the subjective knowledge of the forecaster; a second meteorologist with different knowledge (see the right-hand side of Fig. 1) may issue an alternative forecast.

Moving to quantum mechanics, one may ask whether the wave function Ψ that we assign to a quantum system should be seen as a subjective object (analogous to the weather forecast) representing the knowledge an experimenter has about the system or whether Ψ is an element of reality of the system (analogous to the weather being sunny). This question has been the subject of a long debate, which goes back to the early days of quantum theory [1].

The debate originated from the fact that quantum theory is inherently probabilistic: Even with a full description of a system's wave function, the theory does not allow us to predict the outcomes of future measurements with certainty. This fact is often used to motivate subjective interpretations of quantum theory, such as the Copenhagen interpretation [2–4], according to which wave functions are mere mathematical objects that allow us to calculate probabilities of future events.

Einstein, Podolsky, and Rosen advocated the view that the wave function does not provide a complete physical description of reality [5] and that a higher, complete theory is possible. In such a complete theory, any element of reality must have a counterpart in the theory. Were quantum theory not complete, it could be that the higher theory has additional parameters that complement the wave function. The wave function could then be objective, i.e., uniquely determined by the elements of reality of the higher theory. Alternatively, the wave function could take the role of a state of knowledge about the underlying parameters of the higher theory. In this case, the wave function would not be uniquely determined by these parameters and would therefore admit a subjective interpretation. To connect to some terminology in the literature (see, for example, Ref. [6]), in the first case the underlying model would be called ψ-ontic and in the second case ψ-epistemic. For some recent work in support of a ψ-epistemic view, see, for example, Refs. [7–9].

In some famous works from the 1960s, several constraints were placed on higher descriptions given in terms of hidden variables [10–12], and further constraints have since been highlighted [13–15]. In addition, we have

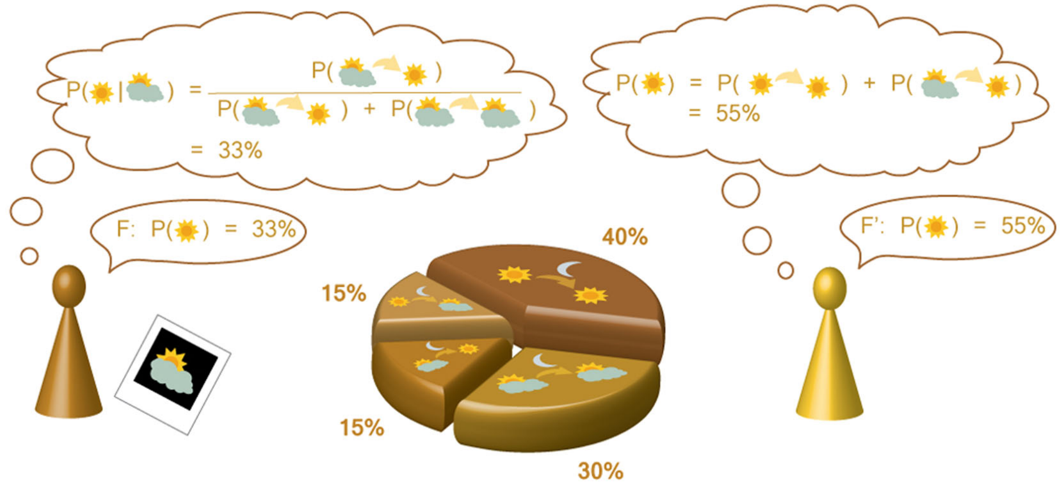

FIG. 1 (color online). Simple example illustrating the ideas. Two meteorologists attempt to predict tomorrow's weather (whether it will be sunny or cloudy in a particular location). Both have access to historical data giving the joint distribution of the weather on successive days. However, only the meteorologist on the left has access to today's weather, and consequently the two make different probabilistic forecasts, F and F'. Assuming that the processes relevant to the weather are accurately described by classical mechanics and thus deterministic, the list of elements of reality, Λ, may include tomorrow's weather, X. Such a list Λ would then necessarily satisfy $\Gamma \leftrightarrow \Lambda \leftrightarrow X$ (for any arbitrary Γ) and therefore be complete [cf. Eq. (1)]. However, the analogue of Eq. (2), $\Lambda \leftrightarrow F \leftrightarrow X$, would imply $X \leftrightarrow F \leftrightarrow X$. This Markov chain cannot hold for the nondeterministic forecasts F and F', which are hence not complete. This is unlike the quantum-mechanical wave function, which gives a complete description for the prediction of measurement outcomes. Note that this difference explains why, in contrast to the quantum-mechanical wave function, F and F' need not be included in Λ and can therefore be considered subjective.

recently shown [16] that, under the assumption of free choice, if quantum theory is correct, then it is *nonextendible*, in the sense of being maximally informative about measurement outcomes.

Very recently, Pusey, Barrett, and Rudolph [17] have presented an argument showing that a subjective interpretation of the wave function would violate certain plausible assumptions. Specifically, their argument refers to a model where each physical system possesses an individual set of (possibly hidden) elements of reality, which are the only quantities relevant for predicting the outcomes of later measurements. One of their assumptions then demands that it is possible to prepare multiple systems such that these sets are statistically independent.

Here, we present a totally different argument to show that the wave function of a quantum system is fully determined by its elements of reality. In fact, this implies that the wave function is in one-to-one correspondence with these elements of reality (see the conclusions) and may therefore itself be considered an element of reality of the system. These claims are derived under minimal assumptions, namely, that the statistical predictions of existing quantum theory are correct and that measurement settings can (in principle) be chosen freely. In terms of the language of Ref. [6], this means that any model of reality consistent with quantum theory and with free choice is ψ-complete.

General model.—In order to state our result, we consider a general experiment where a system $\mathcal{S}$ is prepared in a state specified by a wave function Ψ (see Fig. 2). Then an experimenter chooses a measurement setting A (specified by an observable or a family of projectors) and records the measurement outcome, denoted by X. Mathematically, we model Ψ as a random variable over the set of wave functions, A as a random variable over the set of observables, and X as a random variable over the set of possible measurement outcomes. Finally, we introduce a collection of random variables, denoted by Γ, which are intended to model all information that is (in principle) available before the measurement setting A is chosen and the measurement is carried out. Technically, we require only that Γ includes the wave function Ψ. In the following, when we refer to a list of elements of reality, we simply mean a subset Λ of Γ. Furthermore, we say that Λ is complete for the description of the system $\mathcal{S}$ if any possible prediction about the outcome X of a measurement A on $\mathcal{S}$ can be obtained from Λ; i.e., we demand that the Markov condition

$$\Gamma \leftrightarrow (\Lambda, A) \leftrightarrow X \tag{1}$$

holds [18]. Note that, by using this definition, the aforementioned result on the nonextendibility of quantum theory [16] can be phrased as follows: The wave function Ψ associated with a system $\mathcal{S}$ is complete for the description of $\mathcal{S}$.

We are now ready to formulate our main technical claim.

Theorem.—Any list of elements of reality, Λ, that is complete for the description of a system $\mathcal{S}$ includes the

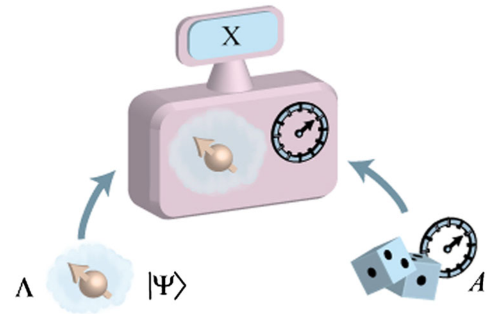

FIG. 2 (color online). Illustration of the setup. A system is prepared in a particular quantum state (specified by a wave function Ψ). The elements of reality, Λ, may depend on this preparation. A measurement setting A is then randomly chosen, and the system is measured, producing an outcome X. We assume that Λ is *complete for the description of the system*, in the sense that there does not exist any other parameter that provides additional information (beyond that contained in Λ) about the outcome of any chosen measurement. In particular, Ψ cannot provide more information than Λ. Conversely, the nonextendibility of quantum theory [16] implies that Λ cannot provide more information (about the outcome) than Ψ. Taken together, these statements imply that Ψ and Λ are informationally equivalent. From this and the fact that different quantum states generally lead to different measurement statistics, we conclude that Ψ must be included in the list Λ and is therefore an element of reality of the system.

quantum-mechanical wave function Ψ associated with $\mathcal{S}$ (in the sense that Ψ is uniquely determined by Λ).

Assumptions.—The above claim is derived under the following two assumptions, which are usually implicit in the literature. (We note that very similar assumptions are also made in Ref. [17], where, as already mentioned, an additional statistical independence assumption is also used.)

(1) *Correctness of quantum theory.*—Quantum theory gives the correct statistical predictions. For example, the distribution of X satisfies $P_{X|\Psi=\psi,A=a}(x) = \langle\psi|\Pi_x^a|\psi\rangle$, where Π_x^a denotes the projector corresponding to outcome $X = x$ of the measurement specified by $A = a$.

(2) *Freedom of choice.*—Measurement settings can be chosen to be independent of any preexisting value (in any frame) [19]. In particular, this implies that the setting A can be chosen independently of Γ, i.e., $P_{A|\Gamma=\gamma} = P_A$ [22].

We note that the proof of our result relies on an argument presented in Ref. [16], where these assumptions are also used (see [21] for more details).

Proof of the main claim.—As shown in Ref. [16], under the above assumptions, Ψ is complete for the description of $\mathcal{S}$. Since Λ is included in Γ, we have, in particular,

$$\Lambda \leftrightarrow (\Psi, A) \leftrightarrow X. \tag{2}$$

Our argument then proceeds as follows. The above condition is equivalent to the requirement that

$$P_{X|\Lambda=\lambda,\Psi=\psi,A=a} = P_{X|\Psi=\psi,A=a}$$

holds for all λ, ψ, and a that have a positive joint probability, i.e., $P_{\Lambda\Psi A}(\lambda, \psi, a) > 0$. Furthermore, because of the assumption that Λ is a complete list of elements of reality, Eq. (1), and because Ψ is by definition included in Γ, we have

$$P_{X|\Lambda=\lambda,\Psi=\psi,A=a} = P_{X|\Lambda=\lambda,A=a}.$$

Combining these expressions gives

$$P_{X|\Psi=\psi,A=a} = P_{X|\Lambda=\lambda,A=a}, \tag{3}$$

for all values λ, ψ, and a with $P_{\Lambda\Psi A}(\lambda, \psi, a) > 0$. Note that, using the free choice assumption, we have $P_{\Lambda\Psi A} = P_{\Lambda\Psi} \times P_A$; hence, this condition is equivalent to demanding $P_{\Lambda\Psi}(\lambda, \psi) > 0$ and $P_A(a) > 0$.

Now consider some fixed $\Lambda = \lambda$ and suppose that there exist two states ψ_0 and ψ_1 such that $P_{\Lambda\Psi}(\lambda, \psi_0) > 0$ and $P_{\Lambda\Psi}(\lambda, \psi_1) > 0$. From Eq. (3), this implies $P_{X|\Psi=\psi_0,A=a} = P_{X|\Psi=\psi_1,A=a}$ for all a such that $P_A(a) > 0$. However, within quantum theory, it is easy to choose the set of measurements for which $P_A(a) > 0$ such that this can be satisfied only if $\psi_0 = \psi_1$. This holds, for example, if the set of measurements is tomographically complete. Thus, for each $\Lambda = \lambda$, there exists only one possible value of $\Psi = \psi$ such that $P_{\Lambda\Psi}(\lambda, \psi) > 0$; i.e., Ψ is uniquely determined by Λ, which is what we set out to prove.

Discussion and conclusions.—We have shown that the quantum wave function can be taken to be an element of reality of a system based on two assumptions: the correctness of quantum theory and the freedom of choice for measurement settings. Both of these assumptions are, in principle, experimentally falsifiable (see [21] for a discussion of possible experiments).

The correctness of quantum theory is a natural assumption given that we are asking whether the quantum wave function is an element of reality of a system. Furthermore, a free choice assumption is necessary to show that the answer is yes. Without free choice, A would be predetermined and the complete list of elements of reality, Λ, could be chosen to consist of the single element X. In this case, Eq. (1) would be trivially satisfied. Nevertheless, since the list $\Lambda = \{X\}$ does not uniquely determine the wave function Ψ, we could not consider Ψ to be an element of reality of the system. This shows that the wave function would admit a subjective interpretation if the free choice assumption were dropped.

We conclude by noting that, given any complete list of elements of reality, Λ, the nonextendibility of quantum theory, Eq. (2), asserts that any information contained in Λ that may be relevant for predicting measurement outcomes X is already contained in the wave function Ψ. Conversely, the result shown here is that Ψ is included in Λ. Since these are two seemingly opposite statements, it is somewhat intriguing that the second can be inferred from the first, as shown in this Letter. Furthermore, taken together, the two statements imply that Ψ is in one-to-one correlation to Λ. This sheds new light on a question dating back to the early days of quantum theory [23], asking whether the wave function is in one-to-one correlation with physical reality. Interpreting Λ as the state of physical reality (or the ontic state), our result asserts that, under the free choice assumption, the answer to this question is yes.

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[1] G. Bacciagaluppi and A. Valentini, *Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference* (Cambridge University Press, Cambridge, England, 2009).
[2] M. Born, Z. Phys. **38**, 803 (1926).
[3] N. Bohr, Naturwissenschaften **16**, 245 (1928).
[4] W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago, Chicago, 1930).
[5] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[6] N. Harrigan and R. W. Spekkens, Found. Phys. **40**, 125 (2010).
[7] C. M. Caves, C. A. Fuchs, and R. Schack, Phys. Rev. A **65**, 022305 (2002).
[8] R. W. Spekkens, Phys. Rev. A **75**, 032110 (2007).
[9] M. S. Leifer and R. W. Spekkens, arXiv:1107.5849.
[10] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[11] J. S. Bell, in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987), Chap. 1.
[12] J. S. Bell, in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987), Chap. 2.
[13] L. Hardy, Stud. Hist. Phil. Mod. Phys. **35**, 267 (2004).
[14] A. Montina, Phys. Rev. A **83**, 032107 (2011).
[15] Z. Chen and A. Montina, Phys. Rev. A **83**, 042110 (2011).
[16] R. Colbeck and R. Renner, Nature Commun. **2**, 411 (2011).
[17] M. F. Pusey, J. Barrett, and T. Rudolph, arXiv:1111.3328.
[18] $U \leftrightarrow V \leftrightarrow W$ is called a Markov chain if $P_{U|V=v} = P_{U|V=v,W=w}$ or, equivalently, if $P_{W|V=v} = P_{W|V=v,U=u}$, for all u, v, and w with strictly positive joint probability. Here and in the following, we use uppercase letters for random variables and lowercase letters for specific values they can take.
[19] This assumption, while often implicit, is, for instance, discussed (and used) in Bell's work. In Ref. [20], he writes that "the settings of instruments are in some sense free variables… [which] means that the values of such variables have implications only in their future light cones." This leads directly to the freedom of choice assumption as formulated here. We refer to [21] for a more detailed discussion.
[20] J. S. Bell, in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987), Chap. 12.
[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.108.150402 for further discussion of the assumptions and for the connection to experiments.
[22] In Ref. [17], this assumption corresponds to the requirement that a quantum system can be freely prepared according to one of a number of predefined states.
[23] Einstein to Schrödinger, 19 June 1935, extracts translated in D. Howard, Stud. Hist. Phil. Sci. **16**, 171 (1985).