Distribution of Particles and Bubbles in Turbulence at a Small Stokes Number

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The inertia of particles driven by the turbulent flow of the surrounding fluid makes them prefer certain regions of the flow. The heavy particles lag behind the flow and tend to accumulate in the regions with less vorticity, while the light particles do the opposite. As a result of the long-time evolution, the particles distribute over a multifractal attractor in space. We consider this distribution using our recent results on the steady states of chaotic dynamics. We describe the preferential concentration analytically and derive the correlation functions of density and the fractal dimensions of the attractor. The results are obtained for real turbulence and are testable experimentally.

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Recently the problem of the distribution of inertial particles in homogeneous turbulence has received a lot of attention from researchers [1–13]. This is largely due to the breakthrough in the theoretical understanding of the Lagrangian motion of particles in the flow that occurred lately [14]. While the understanding of the behavior of particles that have negligible inertia and follow the flow is quite complete by now [14], the understanding of the behavior of inertial particles is still insufficient. Such understanding is especially important because the subject has an extremely wide range of applications: the flows of fluids are typically turbulent and often laden with external particles. Theoretical advancement was made mainly for the case of a small Stokes number (St), where the inertia is weak and the particles "almost" follow the flow. Even in this limit of small Stokes numbers, the particles' distribution is highly nontrivial. Particles' deviations from the surrounding flow accumulate with time, bringing particles to a strange attractor in space. This attractor is multifractal and the only theoretical result obtained so far for the real turbulent flow was the derivation of the correlation codimension [4]. Here a result obtained for real turbulence is a result obtained without modeling turbulence and expressed in terms of the (unknown) statistical properties of turbulence. Since the statistics of turbulence is largely unknown [15], then to obtain such a result, one needs to make universal predictions on particles' behavior in the flow independent of the details of the statistics of that flow.

In this Letter, we provide the complete description of the distribution of particles in real turbulence at small Stokes numbers, describing both the correlation of the particles' density with the surrounding flow and the statistics of the singular density on the attractor. We give a number of predictions that are testable experimentally.

The idea that particles' inertia leads to inhomogeneous spatial distribution dates back to the seminal paper by Maxey [2]. It was observed that due to inertia, heavy particles are pushed out of the vortices and, hence, they will not distribute uniformly in the flow, like the inertialess particles. However, the quantitative description of the correlations between the locations of particles and of vortices stayed unaddressed. Note that the distribution of vorticity in turbulence is random and dynamical, while the distribution of particles reflects its cumulative effect over time. There is a residual correlation that we describe by an integral relation holding in the steady state.

We find the spectrum of fractal dimensions of the attractor. We show that while the correlation dimension is different from the dimension of space, the fractal or similarity dimension [16] is equal to the space dimension. In contrast, the information dimension is different from the spatial dimension and it equals the Kaplan-Yorke dimension. In turn, the correlation codimension equals twice the Kaplan-Yorke codimension which constitutes a prediction allowing direct testing in the laboratory.

The analysis is based on the recent finding of a universal description for the steady state density of the weakly compressible dynamical systems [17]. The particles' motion, though governed by Newton's law, admits an effective description in terms of a velocity field in space. Inertia is described by a small compressible correction to the incompressible velocity of the background turbulent flow. This correction leads to a small imbalance of trajectories going in and out of space regions, which accumulates over a long time to a big effect. Thus, compressibility is a singular perturbation, which treatment was performed in [17]. For a mixing incompressible velocity the evolution of a small volume of particles makes it dense in space. The volume's coarse graining over an arbitrarily small scale covers all the available space, which volume is assumed finite. When a small compressible component is added to the velocity, the coarse graining of the evolved volume over an arbitrarily small scale does not cover the whole space any longer. However, the coarse graining over a small but finite scale, that tends to zero with compressibility, already covers the whole volume.

The analysis assumes the single-particle approximation where one neglects the interaction between the particles and their back reaction on the flow. We consider a small spherical particle with the radius a and the material density ρ_p suspended in a fluid with the density ρ and the kinematic viscosity ν. The fluid flow $\boldsymbol{u}(t,\boldsymbol{x})$ is assumed to be incompressible. The Newton law governing the evolution of the particle's position $\boldsymbol{q}(t)$ and the particle's velocity $\boldsymbol{v}(t)$ is assumed to have the form

$$\frac{d\boldsymbol{v}}{dt} = \gamma \frac{d}{dt}\boldsymbol{u}[t,\boldsymbol{q}(t)] - \frac{\boldsymbol{v} - \boldsymbol{u}[t,\boldsymbol{q}(t)]}{\tau}, \quad (1)$$

where $\gamma = 3\rho/(\rho + 2\rho_p)$ and $\tau = a^2/(3\nu\gamma)$ is the Stokes time. Thus, we assume that all the forces besides the added mass and the drag can be neglected [2,10]. With no loss, we set the total volume and the mass equal to one, so the spatial average of the particles' density n obeys $\langle n\rangle = 1$. We set $\beta = \gamma - 1$ so the particle's velocity relative to the flow $\boldsymbol{w}(t) \equiv \boldsymbol{v}(t) - \boldsymbol{u}[t,\boldsymbol{q}(t)]$ obeys

$$\frac{d\boldsymbol{w}}{dt} = -\frac{\boldsymbol{w}}{\tau} + \beta\frac{d}{dt}\boldsymbol{u}[t,\boldsymbol{q}(t)]. \quad (2)$$

The parameter $\beta = 2(\rho - \rho_p)/(\rho + 2\rho_p)$ changes from -1 for heavy particles to 2 for light ones. After transients

$$\boldsymbol{w}(t) = \beta\int_{-\infty}^{t} \exp\left[-\frac{t-t'}{\tau}\right]\frac{d}{dt'}\boldsymbol{u}[t',\boldsymbol{q}(t')]dt'. \quad (3)$$

We assume τ is much smaller than the smallest time scale of turbulence, which is the viscous time scale t_η, so the Stokes number $\mathrm{St} \equiv \tau/t_\eta \ll 1$ (the Kubo number, measuring velocity correlations in time, is of order one for turbulence [13]). Then we can substitute the derivative in the integrand by its value at $t' = t$ so $\boldsymbol{v}(t) \approx \boldsymbol{u} + \mu[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$ with $\mu \equiv \beta\tau = 2a^2(\rho - \rho_p)/(9\nu\rho)$. Thus at $\mathrm{St} \ll 1$ the particle's velocity is determined uniquely by its position. One can label trajectories $\boldsymbol{q}(t,\boldsymbol{x})$ by their $t = 0$ position $\boldsymbol{x}$ and introduce the particle's flow $\boldsymbol{v}(t,\boldsymbol{x})$,

$$\partial_t\boldsymbol{q}(t,\boldsymbol{x}) = \boldsymbol{v}[t,\boldsymbol{q}(t,\boldsymbol{x})], \quad \boldsymbol{v} \equiv \boldsymbol{u} + \mu[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]. \quad (4)$$

In the zero inertia limit $\mathrm{St} \to 0$ the particles follow the incompressible mixing flow of turbulence $\dot{\boldsymbol{q}} = \boldsymbol{u}[t,\boldsymbol{q}(t)]$ and in the steady state they are uniformly distributed in space, so their steady state density n_s equals one. This behavior is characteristic of small dye particles. However, at a small but finite St, the small correction $\boldsymbol{v}$-$\boldsymbol{u}$ gives the particles' velocity field a finite compressibility [2]

$$w \equiv \nabla\cdot\boldsymbol{v} = -\mu\phi \neq 0, \qquad \phi = \omega^2 - s^2, \quad (5)$$

so the constant is no longer a solution to the continuity equation $\partial_t n + \nabla\cdot(n\boldsymbol{v}) = 0$. Above $s^2 = s_{ij}s_{ij}$ and $\omega^2 = a_{ij}a_{ij}$, where s_{ij} is the symmetric (strain) and a_{ij} is the antisymmetric (vorticity) parts of the velocity gradient $\partial_j u_i = s_{ij} + a_{ij}$. The field $\phi(\boldsymbol{x})$ is positive in the regions dominated by vorticity and negative in the regions dominated by the strain, and it will be called below the indicator, indicating whether $\boldsymbol{x}$ is in a vortex. It follows from the Navier-Stokes equations that ϕ equals the Laplacian of the turbulent pressure, $\phi = \nabla^2 p$. Equation (5) shows that heavy particles $\beta < 0$ are repelled from vortices (here and below "vortex" is used qualitatively), while the light ones $\beta > 0$ are attracted. This is the generalization of the familiar fact that a heavy particle in a centrifuge is pushed out to the boundary. Turbulence can be considered as a dynamically changing spatial distribution of vorticity, so heavy particles tend to accumulate on the boundaries between the vortices cf. [2,4,11], forming a singular density supported on these boundaries. This accumulation however is insignificant during the lifetime of a single vortex and the ultimate singular distribution of particles in space n_s forms from the long-time combined action of many uncorrelated vortices. Still one expects a residual correlation between the distributions of vorticity and particles, to find which, we consider the steady state density n_s. One expects n_s can be obtained by letting an arbitrary initial condition n_0 in the remote past $n(t = -T) = n_0$ evolve for infinite time, $T \to \infty$, according to the continuity equation. Starting from the uniform initial distribution we obtain the steady state density

$$n_s(\boldsymbol{x}) = \lim_{T\to\infty} n(T), \qquad n(T) = \exp\left[-\int_{-T}^{0} w[t,\boldsymbol{q}(t,\boldsymbol{x})]dt\right], \quad (6)$$

if the different-time correlation function of w with an arbitrary function f decays at large times [17]. The decay holds for the mixing turbulence. For the cross correlation of density and vorticity $F(\boldsymbol{x}) = \langle\phi(0)n_s(\boldsymbol{x})\rangle$ we find

$$F(\boldsymbol{x}) = \left\langle \phi(0,0)\exp\left[\mu\int_{-\infty}^{0}\phi[t,\boldsymbol{q}(t,\boldsymbol{x})]dt\right]\right\rangle = \partial_\alpha \ln\left\langle \exp\left[\alpha\phi(0,0) + \mu\int_{-\infty}^{0}\phi[t,\boldsymbol{q}(t,\boldsymbol{x})]dt\right]\right\rangle\Big|_{\alpha=0}, \quad (7)$$

where we used the conservation of the mean density $\langle n(t)\rangle = \mathrm{const}$. Applying the cumulant expansion [18], taking derivative of the series, and setting $\alpha = 0$ we find

$$F(\boldsymbol{x}) = \mu\int_{-\infty}^{0} dt\langle\langle\phi(0,0)\phi[t,\boldsymbol{q}(t,\boldsymbol{x})]\rangle\rangle_c + O(\mathrm{St}^2). \quad (8)$$

The above formula is the same as one would obtain by expanding the exponent in Eq. (7) and keeping the lowest order term in τ, with one important difference. In Eq. (8), one has the second order cumulant or dispersion that one would not get by the series expansion of the exponent cf. [2]. This difference is essential as without the cumulant the integral in Eq. (8) diverges: $\langle w[t,\boldsymbol{q}(t,\boldsymbol{x})]\rangle = (1-\beta)\times\tau\langle\phi[t,\boldsymbol{q}(t,\boldsymbol{x})]\rangle$ is equal to a nonzero sum of Lyapunov exponents evaluated below. To leading order in St, one can substitute $\boldsymbol{q}(t,\boldsymbol{x})$ in Eq. (8) by $\boldsymbol{X}(t,\boldsymbol{x})$

$$\partial_t\boldsymbol{X}(t,\boldsymbol{x}) = \boldsymbol{u}[t,\boldsymbol{X}(t,\boldsymbol{x})], \; \boldsymbol{X}(0,\boldsymbol{x}) = \boldsymbol{x}, \quad (9)$$

where $\boldsymbol{X}(t, \boldsymbol{x})$ are Lagrangian trajectories of $\boldsymbol{u}$. One finds

$$\langle \phi(0) n_s(\boldsymbol{x})\rangle = \mu \int_{-\infty}^0 dt \langle \phi(0,0)\phi[t, \boldsymbol{X}(t,\boldsymbol{x})]\rangle, \quad (10)$$

where we can already omit the cumulant since by incompressibility $\langle w[t, \boldsymbol{X}(t,\boldsymbol{x})]\rangle = \langle w(t,\boldsymbol{x})\rangle$, while $\int w(t,\boldsymbol{x}) d\boldsymbol{x} = \int \nabla\cdot \boldsymbol{v}(t,\boldsymbol{x}) = 0$ by the boundary conditions. Since there is no degeneracy, the non-negative spectrum of the Laplacian of pressure in the Lagrangian frame $\phi[t, \boldsymbol{X}(t,\boldsymbol{x})]$ is strictly positive at zero frequency

$$E(0) = \int_{-\infty}^{\infty} \langle \phi(0,0)\phi[t, \boldsymbol{X}(t,0)]\rangle dt > 0. \quad (11)$$

The single-point correlation $\langle \phi n_s\rangle \equiv \int \phi(\boldsymbol{x}) n_s(\boldsymbol{x}) d\boldsymbol{x}$ equals $\mu E(0)/2$ and it gives the integral of ϕ where each region is weighted by the number of particles in it

$$\int [\omega^2(\boldsymbol{x}) - s^2(\boldsymbol{x})] n_s(\boldsymbol{x}) d\boldsymbol{x} = a^2(\rho - \rho_p)(9\nu\rho)^{-1} E(0), \quad (12)$$

where we used the definitions of ϕ and μ. For heavy particles, $\rho_p > \rho$, the answer is negative giving a measure of the extent to which the particles favor regions with negative ϕ. For light particles, $\rho_p < \rho$, the answer is positive measuring their favoring of vortices. The above integral steady state relation holds at any t.

The quantity $E(0)$ appeared first in [4], where it was shown to determine the correlation dimension of the particles' attractor in space. This quantity is increased by the intermittency of turbulence and it can be estimated as $t_\eta^{-3} f(\text{Re})$ where $f(\text{Re})$ is a growing function of the Reynolds number (Re) that grows as a power [4,5]. We show $E(0)$ determines all the fractal dimensions.

We observe that Eq. (4) is a weakly dissipative dynamical system, defined as the dynamics for which the potential part of $\boldsymbol{v}$ is much smaller than the solenoidal one. The statistics of the steady state density of such systems was shown recently to allow for a complete and universal description [17]. The application to our case gives the following results. The motion of particles in space is chaotic and is characterized by the Lyapunov exponents [14]. To the lowest order in St, the exponents are equal to the Lyapunov exponents λ_i of the turbulent flow $\boldsymbol{u}$. However, the value of the sum of the Lyapunov exponents $\sum \lambda_i^+$ that determines the logarithmic rate of growth of the volumes forward in time,

$$\sum \lambda_i^+(\boldsymbol{x}) \equiv \lim_{t\to\infty} t^{-1} \ln \det \nabla_j q_i(t, \boldsymbol{x}), \quad (13)$$

is zero for $\boldsymbol{u}$, so the leading order approximation demands the account of the correction $\boldsymbol{v}$-$\boldsymbol{u}$. This is also the case of the sum of the Lyapunov exponents $\sum \lambda_i^-$ of the backward-in-time flow that determines the density [17]

$$\lim_{T\to\infty} T^{-1} \ln n[t=0, \boldsymbol{x} | n(t=-T) = n_0] = \sum \lambda_i^-(\boldsymbol{x}). \quad (14)$$

For turbulence $\sum \lambda_i^\pm$ is expected to be the same for all $\boldsymbol{x}$ with the possible exception of a set of points with zero volume. The results of [17,19] give

$$\sum \lambda_i^\pm \approx -\frac{1}{2}\int_{-\infty}^{\infty} \langle w(0,0) w[t, \boldsymbol{X}(t,0)]\rangle = -\frac{\mu^2 E(0)}{2}.$$

Thus for all initial points, with the possible exception of a set of points with zero volume, the infinitesimal volumes decay to zero in the limit of infinite evolution time, while the steady state density n_s is zero except for a set of points with zero volume. Because of conservation of mass $\int n d\boldsymbol{x}$, we conclude that n_s has δ-function type singularities on its support. This support is the strange attractor—the multifractal set in space that is approached by the particles' trajectories at large times. We now find the Kaplan-Yorke codimension C_{KY} of the attractor. At weak compressibility the definition [20] of C_{KY} reduces [17] to $C_{\text{KY}} = \sum \lambda_i^+ / \lambda_3^+$ which gives to the leading order

$$C_{\text{KY}} = \frac{\mu^2 E(0)}{2|\lambda_3|} = \frac{2a^4(\rho - \rho_p)^2 E(0)}{81\nu^2\rho^2|\lambda_3|}, \quad (15)$$

where the third Lyapunov exponent λ_3 determines the rate of exponential separation of $\boldsymbol{X}(t,\boldsymbol{x})$ back in time. We have $|\lambda_3| \sim t_\eta^{-1}$ and $C_{\text{KY}} \sim \beta^2 \text{St}^2 f(\text{Re})$, [4,7]. Despite St^2 dependence, C_{KY} is nonsmall at $\text{St} \ll 1$, [5].

The probability for two particles to be at the distance $\boldsymbol{x}$ is described by the pair-correlation function $\langle n_s(0) n_s(\boldsymbol{x})\rangle$. Substituting the expression from Eq. (6) for n_s and using the cumulant expansion [17] one finds $\langle n_s(0) n_s(\boldsymbol{x})\rangle = \exp[\mu^2 g(\boldsymbol{x})]$ where the structure function $g(\boldsymbol{x})$ depends only on the statistics of turbulence

$$g(\boldsymbol{x}) \equiv \int_{-\infty}^0 dt_1 dt_2 \langle \phi[t_1, \mathbf{X}(t_1, 0)]\phi[t_2, \mathbf{X}(t_2, \mathbf{x})]\rangle. \quad (16)$$

The above is valid if the higher order terms in the cumulant expansion are negligible [17]. The Kolmogorov theory (KT) estimate would give the validity condition $\text{St} \ll 1$, while the account of intermittency changes the condition to $h(\text{Re})\text{St} \ll 1$ where $h(\text{Re})$ is expected to be a slowly growing function of Re cf. [5]. The function $g(\boldsymbol{x})$ has a universal behavior [17] at small $\boldsymbol{x}$ that gives

$$\langle n_s(0) n_s(\boldsymbol{x})\rangle = (\eta/x)^{2C_{\text{KY}}}, \qquad x \ll \eta, \quad (17)$$

where $\eta \sim (\nu t_\eta)^{1/2}$ is the Kolmogorov scale of turbulence [4]. Thus, the correlation codimension equals $2C_{\text{KY}}$.

Remarkably, the structure function determines all the correlation functions of n_s. Generalization of the calculation of $\langle n_s(0) n_s(\boldsymbol{x})\rangle$ gives the log-normal statistics [17] $\langle n_s(\boldsymbol{x}_1) n_s(\boldsymbol{x}_2)\ldots n_s(\boldsymbol{x}_N)\rangle = \exp(\mu^2 \sum_{i>j} g[\mathbf{x}_i - \mathbf{x}_j])$. The density n_s does not have physical meaning and we consider the coarse-grained density n_l,

$$m_l(\boldsymbol{x}) \equiv \int_{|\boldsymbol{x}'-\boldsymbol{x}|<l} n_s(\boldsymbol{x}') d\boldsymbol{x}', \qquad n_l(\boldsymbol{x}) \equiv 3m_l(\boldsymbol{x})/(4\pi l^3),$$

where $m_l(\boldsymbol{x})$ is the mass in a small ball. The limits $l \to 0$ and St $\to 0$ do not commute ($\langle n_l^2 \rangle \sim (\eta/l)^{2C_{\mathrm{KY}}}$)

$$\lim_{l\to 0}\lim_{\mathrm{St}\to 0}\langle n_l^2\rangle = 1, \qquad \lim_{\mathrm{St}\to 0}\lim_{l\to 0}\langle n_l^2\rangle = \infty. \tag{18}$$

For any St > 0 the fluctuations of n_l are large for a sufficiently small l. On the other hand, by $\lim_{\mathrm{St}\to 0}\langle n_l^2\rangle = 1$ one sees that for any fixed $l > 0$ the fluctuations of n_l are small for a sufficiently small St. The coarse-grained density is uniform over scales in which minimal value vanishes with St. Thus turbulence effect on the particles depends on the observer's resolution l: at $2C_{\mathrm{KY}}\ln(\eta/l) \gtrsim 1$ segregation holds, while $2C_{\mathrm{KY}}\ln(\eta/l) \ll 1$—mixing. This is how mixing works effectively for particles on a multifractal. Segregation may also bring physical effects [4].

At St $\ll 1$ there is a scale $L \ll \eta$ over which the density is almost uniform. We note that $m_l(t=0, \boldsymbol{x})$ is equal to the mass contained in the preimage of the ball time t ago, which is an ellipsoid around $\boldsymbol{q}(-t, \boldsymbol{x})$ with the largest axis growing as $l\exp[|\lambda_3 t|]$. At $t_* = |\lambda_3|^{-1}\ln(L/l)$ the ellipsoid has the scale over which the density is uniform, so the mass contained in it is just its volume $4\pi l^3\exp[-\int_{-t_*}^0 w\{t', \boldsymbol{q}(t', \boldsymbol{x})]dt'\}/3$ and we find

$$n_l(\boldsymbol{x}) = \exp\left[-\mu\int_{-|\lambda_3|^{-1}\ln(L/l)}^{0}\phi[t', \boldsymbol{q}(t', \boldsymbol{x})]dt'\right], \tag{19}$$

see [17] for details. The smallness of μ brings the expected conclusion that the statistics of n_l is log-normal

$$\langle n_l^p\rangle = (\eta/l)^{C_{\mathrm{KY}}p(p-1)}, \tag{20}$$

generalizing the result for the integer moments following from the log-normality of the correlation functions. The spectrum of the fractal dimensions $D(\alpha) \equiv \lim_{l\to 0}\ln\langle m_l^{\alpha-1}n_s\rangle/[(\alpha-1)\ln l]$ involves the average with n_s, rather than the spatial average [6,16]. To find it, consider $\langle n_l^{\alpha-1}n_s\rangle = \lim_{T\to\infty}\langle\exp\{-\alpha\int_{-t_*}^0\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]\times dt - \int_{-T}^{-t_*}\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]dt\}\rangle$. Because of St $\ll 1$ the contribution of time-intervals with length t_η is negligible and we may substitute the upper limit in the last integral by $-t_* - t_\eta$ which allows us to perform independent averaging $\langle\exp\{-\alpha\int_{-t_*}^0\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]dt - \int_{-T}^{-t_*-t_\eta}\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]dt\}\rangle \approx \langle\exp\{-\alpha\int_{-t_*}^0\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]dt\}\rangle\langle\exp\{-\int_{-T}^{-t_*-t_\eta}\omega[t, \boldsymbol{q}(t, \boldsymbol{r})]\times dt\}\rangle$. However the last average is equal to one by the conservation of mean density, so $\langle n_l^{\alpha-1}n_s\rangle = \langle n_l^\alpha\rangle$ and

$$D(\alpha) = 3 - C_{\mathrm{KY}}\alpha. \tag{21}$$

Our results generalize to the two-dimensional case, where they can be compared with [6]. Working out the small compressibility limit reproduces our answer. Returning to the three-dimensional case, we observe that the fractal dimensions are close to 3 (we do not consider $\alpha \gg 1$) The fractal dimension of the attractor $D(0)$ coincides with the space dimension 3, which is somewhat counter-intuitive since the volume of the attractor is zero. The information dimension $D(1)$ is equal to the Kaplan-Yorke dimension.

At St $\ll 1$, density inhomogeneities are absent in the inertial range [15]. Then Eq. (17) is a complete description. In contrast, at St ~ 1 the inertial range inhomogeneities are important [9] and Eq. (16), extended to hold asymptotically at St ~ 1, gives a unique access to the inhomogeneities. In KT $g(\boldsymbol{x})$ depends only on x and the mean energy dissipation ϵ so $\langle n_s(0)n_s(\boldsymbol{x})\rangle = \exp[C\mu^2\epsilon^{2/3}x^{-4/3}]$. This prediction describes correctly the model of $\mathbf{v}$ decorrelated in time [3,7,8]. However, for turbulence, simulations [9] show $\ln\langle n_s(0)n_s(\boldsymbol{x})\rangle \propto x^{-10/3}$ at moderate Re where KT is expected to work. Noting $g(\boldsymbol{x}) \sim \tau_x^2\partial_x^4\langle[p(\mathbf{x}) - p(0)]^2\rangle$, where τ_x is the relevant time scale, we suggest the difference has the same origin as the deviations of the pressure scaling from KT [21].

A central result is the analytic description of the preferential concentration by Eq. (12). A single number $E(0)$ completely characterizes the influence of turbulence on the log-normal statistics of density at $r \ll \eta$. Log-normality arises because the steady state density is the cumulative result of the creation of inhomogeneities by many uncorrelated vortices, each of which creates but weak inhomogeneity. The fractal structure at scale l forms relatively fast—within the characteristic time-scale $|\lambda_3|^{-1}\ln(\eta/l)$. The predictions are testable.

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[1] M. R. Maxey and J. J. Riley, Phys. Fluids **26**, 883 (1983).
[2] M. R. Maxey, J. Fluid Mech. **174**, 441 (2006).
[3] E. Balkovsky, G. Falkovich, and A. Fouxon, Phys. Rev. Lett. **86**, 2790 (2001).
[4] G. Falkovich, A. Fouxon, and M. Stepanov, Nature (London) **419**, 151 (2002).
[5] G. Falkovich and A. Pumir, Phys. Fluids **16**, L47 (2004).
[6] J. Bec, K. Gawedzki, and P. Horvai, Phys. Rev. Lett. **92**, 224501 (2004).
[7] J. Chun, D. L. Koch, S. L. Rani, A. Ahluwalia, and L. R. Collins, J. Fluid Mech. **536**, 219 (2005).
[8] J. Bec, M. Cencini, and R. Hillerbrand, Phys. Rev. E **75**, 025301 (2007).
[9] J. Bec, L. Biferale, M. Cencini, A. Lanotte, S. Musacchio, and F. Toschi, Phys. Rev. Lett. **98**, 084502 (2007).
[10] E. Calzavarini, M. Cencini, D. Lohse, and F. Toschi, Phys. Rev. Lett. **101**, 084504 (2008).
[11] P. Olla, Phys. Rev. E **81**, 016305 (2010).
[12] M. Wilkinson, B. Mehlig, and K. Gustavsson, Europhys. Lett. **89**, 50002 (2010).
[13] M. Wilkinson, B. Mehlig, and V. Bezuglyy, Phys. Rev. Lett. **97**, 048501 (2006).
[14] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).

[15] U. Frisch, *Turbulence: the Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).
[16] H. G. E. Hentschel and I. Procaccia, Physica D (Amsterdam) **8**, 435 (1983).
[17] I. Fouxon, arXiv:1110.1625.
[18] S.-K. Ma, *Statistical Mechanics* (World Scientific Publishing Company, Singapore, 1984).
[19] G. Falkovich and A. Fouxon, New J. Phys. **6**, 50 (2004).
[20] J. L. Kaplan and J. A. Yorke, *Functional Differential Equations and Approximations of Fixed Points*, edited by H. O. Peitgen and H. O. Walther (Springer, New York, 1979).
[21] T. Gotoh and D. Fukayama, Phys. Rev. Lett. **86**, 3775 (2001).