Integrable Spin Chain for the $\mathrm{SL}(2, R)/\mathrm{U}(1)$ Black Hole Sigma Model

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We introduce a spin chain based on finite-dimensional spin-1/2 SU(2) representations but with a non-Hermitian ''Hamiltonian'' and show, using mostly analytical techniques, that it is described at low energies by the $\mathrm{SL}(2, R)/\mathrm{U}(1)$ Euclidian black hole conformal field theory. This identification goes beyond the appearance of a noncompact spectrum; we are also able to determine the density of states, and show that it agrees with the formulas in [J. Maldacena, H. Ooguri, and J. Son, J. Math. Phys. (N.Y.) **42**, 2961 (2001)] and [A. Hanany, N. Prezas, and J. Troost, J. High Energy Phys. 04 (2002) 014], hence providing a direct ''physical measurement'' of the associated reflection amplitude.

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Introduction.—The profound relation between quantum spin chains and quantum field theories (QFTs) is central to modern theoretical physics. Its simplest aspect is that the low-energy excitations of a chain are described by a QFT in the continuum limit. Conversely, some difficult, strongly interacting QFTs can be tackled using appropriate (usually antiferromagnetic) spin chains, for which a large variety of methods—including numerical—are available. The numerous success stories using this approach include understanding the θ term in the O(3) sigma model [1], and developing bosonization techniques from the concept of Luttinger liquid [2].

The above examples all involve spin chains built with finite-dimensional representations, and their continuum limits are QFTs with a compact target. But many current problems of physics are concerned with strongly curved, noncompact targets. For instance, the conformal field theory (CFT) describing the transition between plateaux in the two-dimensional (2D) integer quantum Hall effect (IQHE) is expected to be the low-energy limit of the noncompact 2D super sigma model on $\mathrm{U}(1,1|2)/[\mathrm{U}(1|1)\times\mathrm{U}(1|1)]$ at $\theta=\pi$. Also, the dual of $\mathcal{N}=4$ SUSY gauge theory in 4D is closely related with a 2D sigma model on $\mathrm{PSU}(2,2|4)/[\mathrm{SO}(4,1)\times\mathrm{SO}(5)]$ [3]. While it seems extremely hard to solve these sigma models directly, one might hope that spin chain regularizations provide access to some of their properties. *A priori*, these chains should involve infinite-dimensional representations. In the IQHE such a chain indeed arises in the very anisotropic limit of the Chalker-Coddington network model, and involves alternating highest and lowest weight representations of $psl(2|2)$ [4].

Unfortunately, the technical difficulties encountered in the analysis of these infinite-dimensional spin chains are considerable. While numerical methods based on Hilbert space truncations are possible [5], analytical approaches have stalled. Despite much work on different aspects of the Bethe ansatz (BA) in this case [6,7], it is not even known whether the antiferromagnetic noncompact *XXX* spin chains are gapless—nor to what extent analyses based on coherent state representations and analogies with the compact case [8] make sense.

In this Letter we show how to construct a solvable, finite-dimensional, antiferromagnetic spin chain whose low-energy physics is described nevertheless by a noncompact CFT. This is obviously important progress, since the usual BA techniques can then be used, without insurmountable difficulties, to understand noncompact CFTs.

We illustrate this discovery with the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ sigma model, a gauged Wess-Zumino-Witten (WZW) model originally introduced in the context of black holes in string theory [9] and later intensively studied for its CFT features as well [10–15]. While it is tempting to assume that it [or the $\mathrm{SL}(2,\mathbb{R})$ WZW model] is the continuum limit of a spin chain based on infinite-dimensional representations of $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$, this connection remains presently entirely speculative. In contrast, we here show how all the known features of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ sigma model, including the noncompact spectrum and the highly nontrivial density of states, are obtainable starting from a rather modest-looking spin chain, one of whose aspects, however, is *non-Hermiticity.*

The spin chain.—The starting point is a $\mathbb{Z}_2$ staggered model, which was introduced in relation with the antiferromagnetic Potts model [16–18]. It is a variant of the six-vertex model but with alternating spectral parameters $(u, u+\frac{\pi}{2},\ldots,u,u+\frac{\pi}{2})$ and $(0,\frac{\pi}{2},\ldots,0,\frac{\pi}{2})$ on the horizontal and vertical lines of the square lattice, respectively. Here, we use the Boltzmann weights: $a=\sin(\gamma-u)$, $b=\sin u$, $c=\sin\gamma$, in Baxter's notations [19], encoded in the matrix $R_{ij}(u)$, and we restrict to the regime $0 < \gamma < \frac{\pi}{2}$. The one-row transfer matrix with periodic boundary conditions, for a system of width $2L$, is

$$t(u) := \mathrm{Tr}_0 R_{0,2L}(\bar{u}) R_{0,2L-1}(u) \dots R_{02}(\bar{u}) R_{01}(u),$$

where $\bar{u} := u - \pi/2$. For simplicity, we suppose L is even. In the very anisotropic limit $u \to 0$, the equivalent quantum Hamiltonian is

$$H := \frac{1}{2} \sin 2\gamma \left[t^{-1}(0) t'(0) + t^{-1}\left(\frac{\pi}{2}\right) t'\left(\frac{\pi}{2}\right) \right], \quad (1)$$

where the prime denotes differentiation with respect to u. In terms of Pauli matrices,

$$H = \sum_{j=1}^{2L} \left[-\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \sin^2\gamma \sigma_j^z \sigma_{j+1}^z - \frac{1}{2} \cos 2\gamma \mathbb{1} + i \sin\gamma (\sigma_{j-1}^z - \sigma_{j+2}^z)(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) \right]. \quad (2)$$

The parameter γ defines the quantum algebra $\mathrm{U}_q(\mathrm{S}\ell_2)$ for the R matrix, through the relation $q = e^{i\gamma}$. Like in the open XXZ chain, this makes H non-Hermitian. However, the low-lying states we study here all have real energies. Among the conserved quantities of H, one has (i) the $\mathbb{Z}_2$ charge $C := \prod_{j=1}^{L} c_{2j-1,2j}$, (ii) the total magnetization $M := \frac{1}{2}\sum_{j=1}^{2L} \sigma_j^z$, and (iii) the "quasimomentum" $S := \frac{\pi - 2\gamma}{4\pi\gamma} \log[t(0) t^{-1}(\frac{\pi}{2})]$, which reads

$$S = \frac{\pi - 2\gamma}{4\pi\gamma} \log\left(\prod_{j=1}^{L} c_{2j,2j+1} \prod_{j=1}^{L} c_{2j-1,2j} \right). \quad (3)$$

We have defined $c_{ij} = P_{ij} R_{ij}(-\pi/2)/\cos\gamma$, and P_{ij} permutes the spins i and j. The three above operators commute with H by the Yang-Baxter equations and the six-vertex "ice rule." Below we derive the low-energy spectrum of (2), and establish a dictionary between the above conserved quantities and those of the $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ sigma model.

Low-energy spectrum from the Bethe ansatz.—The model (2) is solvable by the Bethe ansatz, and the low-energy states correspond to two sets of real roots $\{\lambda_j\}_{j=1\dots r_0}$ and $\{\mu_j\}_{j=1\dots r_1}$. The BA equations read [20]

$$\left[\frac{\cosh(\lambda_j - i\gamma)}{\cosh(\lambda_j + i\gamma)}\right]^L = -\prod_{\ell=1}^{r_0} \frac{\sinh\frac{1}{2}(\lambda_j - \lambda_\ell - 2i\gamma)}{\sinh\frac{1}{2}(\lambda_j - \lambda_\ell + 2i\gamma)} \times \prod_{\ell=1}^{r_1} \frac{\cosh\frac{1}{2}(\lambda_j - \mu_\ell - 2i\gamma)}{\cosh\frac{1}{2}(\lambda_j - \mu_\ell + 2i\gamma)}, \quad (4)$$

$$\left[\frac{\cosh(\mu_j - i\gamma)}{\cosh(\mu_j + i\gamma)}\right]^L = -\prod_{\ell=1}^{r_0} \frac{\cosh\frac{1}{2}(\mu_j - \lambda_\ell - 2i\gamma)}{\cosh\frac{1}{2}(\mu_j - \lambda_\ell + 2i\gamma)} \times \prod_{\ell=1}^{r_1} \frac{\sinh\frac{1}{2}(\mu_j - \mu_\ell - 2i\gamma)}{\sinh\frac{1}{2}(\mu_j - \mu_\ell + 2i\gamma)}, \quad (5)$$

and the corresponding energy and momentum are

$$E = -\sum_{j=1}^{r_0} \frac{2\sin^2 2\gamma}{\cosh 2\lambda_j + \cos 2\gamma} - \sum_{j=1}^{r_1} \frac{2\sin^2 2\gamma}{\cosh 2\mu_j + \cos 2\gamma},$$

$$p = -i \sum_{j=1}^{r_0} \log\frac{\cosh(\lambda_j - i\gamma)}{\cosh(\lambda_j + i\gamma)} - i \sum_{j=1}^{r_1} \log\frac{\cosh(\mu_j - i\gamma)}{\cosh(\mu_j + i\gamma)}.$$

Bethe states are eigenstates of M and S, with eigenvalues

$$m = L - r_0 - r_1, \qquad s = \sum_{j=1}^{r_0} s(\lambda_j) - \sum_{j=1}^{r_1} s(\mu_j),$$

where

$$s(\lambda) := \frac{\pi - 2\gamma}{4\pi\gamma} \log\frac{\cosh\lambda + \sin\gamma}{\cosh\lambda - \sin\gamma}.$$

The operator C exchanges $\{\lambda_j\}$ and $\{\mu_j\}$. Note that, when $\{\lambda_j\} = \{\mu_j\}$, (4) and (5) reduce to the BA of an XXZ model with anisotropy $\Delta = -\cos\gamma'$, where $\gamma' := \pi - 2\gamma$.

In the limit $r_0 = r_1 = L/2 \to \infty$, the roots that solve (4) and (5) form a pair of continuous distributions (η_0, η_1), defined respectively on the intervals $[-\Lambda_0', \Lambda_0]$ and $[-\Lambda_1', \Lambda_1]$, and subject to two coupled linear integral equations:

$$2\pi\eta_a(\lambda) + \sum_{b=0,1} \int_{-\Lambda_b'}^{+\Lambda_b} d\mu\, \eta_b(\mu) K_{a-b}(\lambda - \mu) = \phi(\lambda), \quad (6)$$

where $a \in \{0, 1\}$. It is convenient to define the kernels K_{a-b} and the source term ϕ through their Fourier transform, with the convention $\hat{f}(\omega) := \int d\lambda f(\lambda) e^{i\omega\lambda}$. One has:

$$\hat{K}_0, \hat{K}_{\pm 1}, \hat{\phi} = -\frac{2\pi \sinh\gamma'\omega}{\sinh\pi\omega}, \frac{2\pi \sinh 2\gamma\omega}{\sinh\pi\omega}, \frac{2\pi\sinh\gamma\omega}{\sinh\frac{\pi\omega}{2}}.$$

Like in the XXZ model, the ground state (gs) corresponds to the limit $\Lambda_a = \Lambda_a' \to \infty$ in (6), and the solution is simply obtained by Fourier transform $\eta_0(\lambda) = \eta_1(\lambda) = \eta_{\mathrm{gs}}(\lambda) := 1/(2\gamma' \cosh\frac{\pi\lambda}{\gamma'})$. The central charge obtained from the scaling of the ground-state energy is $c = 2$.

The elementary excitations over the ground state (spinons) are holes in the root distributions η_0, η_1. Using standard kernel methods [21,22], we get the dressed magnetic charge $\mathcal{Z} = \pi/(4\gamma)$ and the energy and momentum of a spinon of rapidity λ:

$$\epsilon_{\mathrm{sp}}(\lambda) = -\frac{\pi\sin\gamma'}{\gamma' \cosh\frac{\pi\lambda}{\gamma'}}, \qquad p_{\mathrm{sp}}(\lambda) = 2\arctan\left(\tanh\frac{\pi\lambda}{2\gamma'}\right).$$

The low-energy spinons ($\lambda \to \infty$) thus have a linear dispersion, with Fermi velocity $v_f = \frac{\pi\sin\gamma'}{\gamma'}$. Similarly, the quasimomentum associated to a spinon is

$$s_{\rm sp}(\lambda) = \pm\frac{\pi - 2\gamma}{4\pi\gamma}\log\left[\cosh\frac{\pi\lambda}{\gamma'}\right],$$

where the sign depends on which distribution (η_0 or η_1) the spinon lives in. Since $s \propto (r_1 - r_0)$ for large L [see (14)], this quantity measures the difference between the two total root densities.

The main object of this Letter is the study of the conformal spectrum $\{(h, \bar{h})\}$ through the finite-size behavior of the energy gap and the momentum [23]:

$$\Delta E = E - E_{\rm gs} \simeq \frac{2\pi v_f}{L}(h + \bar{h}), \qquad p \simeq \frac{2\pi}{L}(h - \bar{h}).$$

In analogy with the XXZ case [24], we assume that the Bethe states which converge to primaries in the continuum limit are combinations of a magnetic excitation (removal of m_0 roots $\{\lambda_j\}$ and m_1 roots $\{\mu_j\}$) and an electric excitation (global shift of the Bethe integers by an integer e): such a state is then denoted $\Psi_{m_0,m_1,e}$. It is well known how to extract conformal weights from the BA equations of a gapless spin chain using Wiener-Hopf (WH) analysis [21]. Reproducing the standard calculation [22] leads immediately to the conformal weights for $\Psi_{m_0,m_1,e}$:

$$h + \bar{h} = \frac{m^2}{8[1 + \hat{\underline{J}}_0(0)]} + 2[1 + \hat{\underline{J}}_0(0)]e^2 + \frac{\tilde{m}^2}{8[1 + \hat{\underline{J}}_1(0)]}, \tag{7}$$

where we introduced the symmetric and antisymmetric magnetic charges $m := m_0 + m_1$ and $\tilde{m} := m_0 - m_1$, and the inverse kernels

$$1 + \hat{\underline{J}}_r(\omega) := \frac{2\pi}{2\pi + \hat{K}_0(\omega) + (-1)^r\hat{K}_1(\omega)}. \tag{8}$$

However, the particular feature of the $\mathbb{Z}_2$ staggered model is that the kernel $(1 + \hat{\underline{J}}_1)$ has a double pole at $\omega = 0$. This means that, for finite $\tilde{m}$, the third term in (7) vanishes identically, suggesting an infinitely degenerate ground state in the thermodynamic limit, or, more accurately, a continuous spectrum of exponents [18]. To establish this, more analysis is obviously needed [25].

We now show how to handle this problem, emphasizing the main differences from [21,22]. Starting from (6), our strategy consists in expanding m, s, $\tilde{m}$, and ΔE in terms of the small parameters $\xi_a := e^{-\pi\Lambda_a/\gamma'}$ and then (as in [21]) eliminating the ξ_a's from the equations, to get ΔE and $\tilde{m}$ as functions of (m, s, L). The finite-size effects on ΔE then yield the conformal weights, whereas the constraint $\tilde{m} \in \mathbb{Z}$ determines the density of states.

We restrict for simplicity to a purely magnetic state $\Psi_{m_0,m_1,0}$, for which the η_a's are even functions. Defining the combinations $\underline{\eta}_r := \eta_0 + (-1)^r\eta_1$, we may rewrite (6) as a pair of coupled WH equations:

$$\underline{\eta}_r(\lambda) + \sum_{a=0,1}(-1)^{ar}\int_{\Lambda_a}^{\infty}\eta_a(\mu)\underline{J}_r(\lambda - \mu)d\mu = 2\delta_{r,0}\eta_{\rm gs}(\lambda). \tag{9}$$

We write the WH decomposition of the kernels as $1 + \hat{\underline{J}}_r(\omega) = [\hat{\underline{G}}_r^+(\omega)\hat{\underline{G}}_r^-(\omega)]^{-1}$, where $\hat{\underline{G}}_r^+$ ($\hat{\underline{G}}_r^-$) is analytic and nonzero in the upper (lower) half-plane. This is given by

$$\begin{aligned}\hat{\underline{G}}_0^+(\omega) &= \frac{\sqrt{4\gamma}\,\Gamma(1 - \frac{i\omega}{2})}{\Gamma(1 - \frac{i\gamma\omega}{\pi})\Gamma(\frac{1}{2} - \frac{i\gamma'\omega}{2\pi})},\\ \hat{\underline{G}}_1^+(\omega) &= \frac{\sqrt{\frac{\gamma\gamma'}{\pi}}\,i\omega\,\Gamma(\frac{1}{2} - \frac{i\omega}{2})}{\Gamma(1 - \frac{i\gamma\omega}{\pi})\Gamma(1 - \frac{i\gamma'\omega}{2\pi})}\end{aligned} \tag{10}$$

and $\hat{\underline{G}}_r^-(\omega) := \hat{\underline{G}}_r^+(-\omega)$. We define the shifted densities $g_a^+(\lambda) := \eta_a(\lambda + \Lambda_a)\Theta(\lambda)$, where Θ stands for Heaviside's step function. The solution of (9) can be expanded on the poles $\{\omega_0, \omega_1, \ldots\}$ of $\hat{\eta}_{\rm gs}$ in the lower half-plane, and the leading order is

$$\hat{g}_a^+(\omega) \simeq \frac{C}{\omega - \omega_0}\sum_{b=0,1}e^{i\omega(\Lambda_b - \Lambda_a)}\hat{G}_{a-b}^+(\omega)\xi_b, \tag{11}$$

where $\hat{G}_{a-b}^+ := \frac{1}{2}[\hat{\underline{G}}_0^+ + (-1)^{a-b}\hat{\underline{G}}_1^+]$, $\omega_0 := -i\pi/\gamma'$ and $C := \hat{\underline{G}}_0^-(\omega_0)\mathrm{Res}(\hat{\eta}_{\rm gs}, \omega_0)$. Following [21], we get:

$$\frac{m}{L} \simeq -\frac{2C(\xi_0 + \xi_1)}{\omega_0\hat{\underline{G}}_0^-(0)}, \qquad \frac{\Delta E}{L} \simeq 2\pi v_f\frac{C^2}{\omega_0^2}(\xi_0^2 + \xi_1^2). \tag{12}$$

The derivation of $\tilde{m}$ and s is more involved, due to the singularity of $\hat{\underline{J}}_1$. From (6), we have

$$\frac{\tilde{m}}{L} = \lim_{\omega\to 0}\frac{\sum_{a=0,1}(-1)^a[e^{i\omega\Lambda_a}\hat{g}_a^+(\omega) + e^{-i\omega\Lambda_a}\hat{g}_a^+(-\omega)]}{\hat{\underline{G}}_1^+(\omega)\hat{\underline{G}}_1^-(\omega)},$$

$$\frac{s}{L} = -\sum_{a=0,1}\frac{(-1)^a}{\pi}\int d\omega\,\hat{s}_{\rm sp}(\omega)\hat{g}_a^+(\omega)e^{i\omega\Lambda_a}.$$

Inserting the WH solution (11) yields

$$\tilde{m} \simeq \frac{2iC(\Lambda_0\xi_0 - \Lambda_1\xi_1)L}{\omega_0(\hat{\underline{G}}_1^-)'(0)}, \qquad s \simeq \frac{\gamma'^2C(\xi_0 - \xi_1)L}{2\pi(\hat{\underline{G}}_1^-)'(0)}. \tag{13}$$

Consider the regime where m and s are finite. Equations (12) and (13) then give the scaling of ξ_0 and ξ_1:

$$\xi_0, \xi_1 \propto \frac{1}{L}, \qquad (\xi_0 - \xi_1) \propto \frac{1}{L}.$$

Eliminating these variables from Eqs. (12) and (13), we obtain

$$\tilde{m} \simeq \frac{4s}{\pi}\left[\log\frac{L}{L_0} + B(\gamma, m, e, s)\right], \tag{14}$$

$$\Delta E \simeq \frac{2\pi v_f}{L}\left(\frac{\gamma m^2}{2\pi} + \frac{2\pi s^2}{\pi - 2\gamma}\right). \tag{15}$$

In Eqs. (14) and (15) (derived here for $e = 0$), L_0 is a cutoff depending only on γ, and B is a correction term which we discuss below. More generally, for $e \neq 0$, a similar derivation yields

$$\Delta E \simeq \frac{2\pi v_f}{L}\left(\frac{\gamma m^2}{2\pi} + \frac{\pi e^2}{2\gamma} + \frac{2\pi s^2}{\pi - 2\gamma}\right), \qquad p = \frac{2\pi e m}{L},$$

and hence

$$h = \frac{(m+ek)^2}{4k} + \frac{s^2}{k-2}, \qquad \bar{h} = \frac{(m-ek)^2}{4k} + \frac{s^2}{k-2}, \tag{16}$$

where $(m, e) \in \mathbb{Z}^2$, $s \in \mathbb{R}$, and we have set $k := \pi/\gamma$. Since s is real, this is a noncompact spectrum.

Relation to the sigma model.—Recall now that, while the $SL(2, \mathbb{R})/U(1)$ black hole (BH) model has a central charge $c_{BH} = 2(k+1)/(k-2)$, the identity field in that theory is associated with a non-normalizable state. In fact, normalizable states arise mostly from continuous representations, and have conformal weights as in (16), but with the second term $s^2/(k-2)$ replaced by a WZW-type term $-j(j+1)/(k-2)$, with $j = -\frac{1}{2} + is$. The "bottom" of the spectrum thus occurs at $h_0 := 1/[4(k-2)]$, leading to an effective central charge $c = c_{BH} - 24h_0 = 2$ as in our lattice model. The spectrum (16) is thus formally identical with the $SL(2, \mathbb{R})/U(1)$ one [11,15].

Since, in the large-L limit, s becomes a real parameter, the spectrum (16) is a collection of continua over the conformal weights of a compact boson. In the $SL(2, \mathbb{R})/U(1)$ theory, this boson describes excitations along the compact direction of the cigar (angular momentum of rotations around the tip), whereas s is the angular momentum along the axis of the cigar. We have expressed in (3) the lattice operator S measuring this angular momentum. In finite size, since $s \simeq \pi\tilde{m}/(4\log L)$, the s^2 terms in (16) correspond to the magnetic charge of a boson with effective compactification radius $R \propto \log L$.

As in ordinary quantum mechanics, there is actually little dynamical information in the spectrum (16) alone: what is really needed is the density of states. This can also be extracted from our finite-size calculation. Denoting $q = \exp(2i\pi\tau)$ the modular parameter, the partition function of our model on a torus reads, in the scaling limit,

$$Z = \frac{(q\bar{q})^{-2/24}}{|\eta(\tau)|^4} \sum_{e\in\mathbb{Z}, m+\tilde{m}\in 2\mathbb{Z}} q^h \bar{q}^{\bar{h}} = \frac{(q\bar{q})^{-2/24}}{|\eta(\tau)|^4} \sum_{e,m\in\mathbb{Z}^2} \int_{-\infty}^{+\infty} ds\,\rho(s) q^h \bar{q}^{\bar{h}},$$

where η is the Dedekind eta function, and the density of states is

$$\rho(s) = \frac{2}{\pi}\left[\log\frac{L}{L_0} + \partial_s(sB)\right], \tag{17}$$

where B was introduced in (14). The logarithmic divergence with the IR cutoff is familiar in the sigma model [15], whereas the finite part of $\rho(s)$ is determined by requiring $\tilde{m} \in \mathbb{Z}$ in (14). Consider the purely magnetic state $\Psi_{m_0,m_1,0}$. Our WH technique only gives access [26] to the function B in the regime of large s and m, where we get

$$B(\gamma, m, e=0, s) \sim \begin{cases} -\log s & \text{for } s \gg m \\ -\log m & \text{for } s \ll m. \end{cases} \tag{18}$$

We believe it will eventually be possible to obtain more complete results on B by a deeper analysis of the Bethe ansatz equations. For now, in order to interpolate between the above limiting behaviors, we compute B numerically, by solving (4) and (5) at finite L: see results on Figs. 1 and 2. The only adjustable parameter in these computations is L_0, which can be fixed, e.g., by imposing the value of B in the ground state $m = e = s = 0$. Slow convergence with the system size is to be expected, because higher-order corrections to Eq. (14) are of order $1/\log L$.

The finite part of the density of states $\rho(s)$ in the $SL(2, \mathbb{R})/U(1)$ sigma model was calculated in [10,11] (see also [27]), and reads, in our parametrization,

$$B_{BH}(s) = \frac{1}{2s}\,\mathrm{Im}\log\left[\Gamma\left(\frac{1-m+ek}{2} - is\right) \times \Gamma\left(\frac{1-m-ek}{2} - is\right)\right]. \tag{19}$$

This function obeys the asymptotic behavior (18), and numerical agreement with the finite part in our model is good, as shown in Figs. 1 and 2. Moreover, we have computed the values of $(B - B_{gs})$, where B_{gs} stands for the ground-state value of B, in the limit $s \to 0$, to check that L_0 depends only on γ: see Fig. 3.

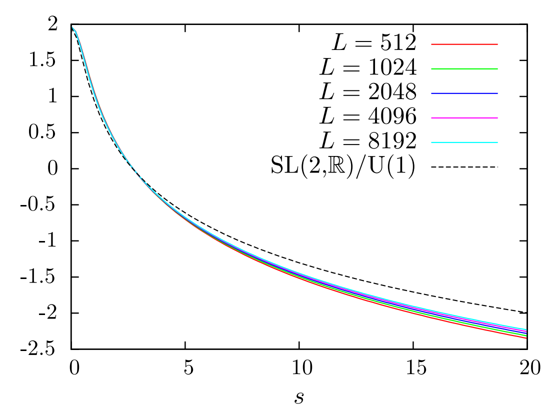

FIG. 1 (color online). Finite part $B(s)$ of the density of states (17) for the continuum over the ground state of the $\mathbb{Z}_2$ model at $\gamma = \pi/5$, compared to B_{BH}.

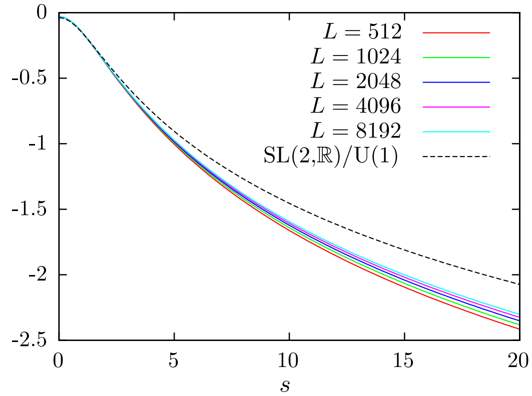

FIG. 2 (color online). Same as Fig. 1, but for the continuum over the excited state ($m = 2$, $e = 0$).

To conclude, we have identified the continuum limit of our spin chain as the $SL(2,\mathbb{R})/U(1)$ black hole sigma model CFT [28], with the level $k \in]2, \infty[$. Obviously, this identification opens the way to much further development. On the one hand, the spin chain can be used to better understand the CFT structures, investigate issues such as discrete states, conformal boundary conditions, etc.—it will be particularly useful to study the so called Destri-de Vega equations in this context [29]. On the other hand, this example is not unique, since there exist [30,31] other spin chains with finite representations and a noncompact continuum limit. Hence, we plan, in particular, to study sigma models with more complicated (super) targets (e.g., for the IQHE plateau theory) using this strategy.

Spin chains have also appeared from a different viewpoint in the AdS/CFT conjecture [32]. It was discovered that many physical quantities on the gauge theory side can be related with the spectra of quantum spin chains [33]. These spectra in turn can be studied by techniques directly addressing the low-energy excitations [34], or via the BA.

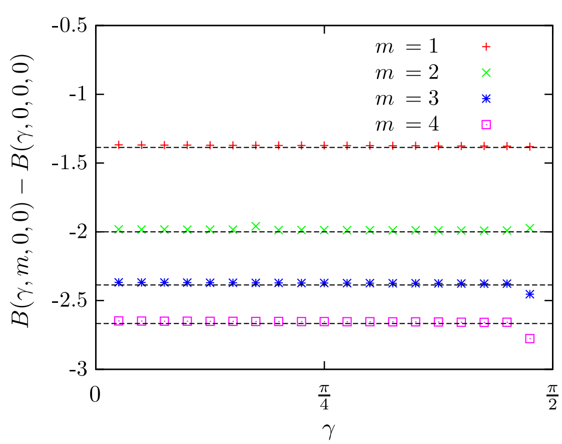

FIG. 3 (color online). The values of B at $s = 0$ along the interval $0 < \gamma < \frac{\pi}{2}$. The data points represent extrapolated numerical values, and the dotted lines are the corresponding values in the $SL(2,\mathbb{R})/U(1)$ model, taken from (19).

Recently, powerful machinery has been developed along those lines to obtain results for the gauge theory at any coupling [35]. It is tempting to conjecture that spin chains such as ours might appear in this context as the "gauge theory" side of some new interesting CFTs.

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