Turbulence in Noninteger Dimensions by Fractal Fourier Decimation

Uriel Frisch,[1] Anna Pomyalov,[2] Itamar Procaccia,[2] and Samriddhi Sankar Ray[1]

[1]*UNS, CNRS, OCA, Laboratoire Lagrange, Boîte Postale 4229, 06304 Nice Cedex 4, France*
[2]*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
(Received 5 August 2011; published 13 February 2012)

Fractal decimation reduces the effective dimensionality D of a flow by keeping only a (randomly chosen) set of Fourier modes whose number in a ball of radius k is proportional to k^D for large k. At the critical dimension $D_c = 4/3$ there is an equilibrium Gibbs state with a $k^{-5/3}$ spectrum, as in V. L'vov *et al.*, Phys. Rev. Lett. **89**, 064501 (2002). Spectral simulations of fractally decimated two-dimensional turbulence show that the inverse cascade persists below $D = 2$ with a rapidly rising Kolmogorov constant, likely to diverge as $(D - 4/3)^{-2/3}$.

In theoretical physics a number of results have been obtained by extending the dimension d of space from directly relevant values such as 1, 2, 3 to noninteger values. Dimensional regularization in field theory [1] and the $4 - \epsilon$ expansion in critical phenomena [2] are well-known instances. For this, one usually expands the solution in terms of Feynman diagrams, each of which can be analytically continued to real or complex values of d. The same kind of extension can be carried out for homogeneous isotropic turbulence but a severe difficulty appears then for $d < 2$: the energy spectrum $E(k)$ can become negative in some band of wave numbers k, so that this kind of extension lacks probabilistic realizability [3]. Nevertheless, in Ref. [4], henceforth cited as LPP, it is argued that, should there exist an alternative realizable way of doing the extension below dimension two in which the nonlinearity conserves energy and enstrophy, then an interesting phenomenon—to which we shall come back—should happen in dimension 4/3.

For diffusion and phase transitions there is a very different way of switching to noninteger dimensions, namely, to reformulate the problem on a fractal of dimension D (here a capital D will always be a fractal dimension) [5]. Are we able to do this for *hydrodynamics*? Implementing mass and momentum conservation on a fractal is quite a challenge [6]. We discovered a new way of fractal decimation in Fourier space, appropriate for hydrodynamics. Since, here, we are primarily interested in dimensions less than two, we shall do our decimation starting from the standard $d = 2$ case.

The forced incompressible Navier-Stokes equations for the velocity field can be written in abstract notation as

$$\partial_t \boldsymbol{u} = B(\boldsymbol{u}, \boldsymbol{u}) + \boldsymbol{f} + \Lambda \boldsymbol{u}, \qquad (1)$$

$$B(\boldsymbol{u}, \boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p, \qquad \Lambda = \nu \nabla^2, \qquad (2)$$

where $\boldsymbol{u}$ stands for the velocity field $\boldsymbol{u}(x_1, x_2, t)$, $\boldsymbol{f}$ for the force $\boldsymbol{f}(x_1, x_2, t)$, p is the pressure and ν the viscosity. The velocity $\boldsymbol{u}$ is taken in the space of divergenceless velocity

fields which are 2π periodic in x_1 and x_2, such that $\boldsymbol{u}(t = 0) = \boldsymbol{u}_0$. Now, we define a Fourier decimation operator P_D on this space of velocity fields:

$$\text{If} \quad \boldsymbol{u} = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} e^{i\boldsymbol{k} \cdot \boldsymbol{x}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}, \quad \text{then} \quad P_D \boldsymbol{u} = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} e^{i\boldsymbol{k} \cdot \boldsymbol{x}} \theta_{\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{k}}. \quad (3)$$

Here, $\theta_{\boldsymbol{k}}$ are random numbers such that

$$\theta_{\boldsymbol{k}} = \begin{cases} 1 & \text{with probability } h_k \\ 0 & \text{with probability } 1 - h_k, \end{cases} \quad k \equiv |\boldsymbol{k}|. \quad (4)$$

To obtain D-dimensional dynamics we choose

$$h_k = C(k/k_0)^{D-2}, \qquad 0 < D \le 2, \qquad 0 < C \le 1, \quad (5)$$

where k_0 is a reference wave number; here $C = k_0 = 1$. All the $\theta_{\boldsymbol{k}}$ are chosen independently, except that $\theta_{\boldsymbol{k}} = \theta_{-\boldsymbol{k}}$ to preserve Hermitian symmetry. Our fractal decimation procedure removes at random—but in a time-frozen (quenched) way—many modes from the $\boldsymbol{k}$ lattice, leaving on average $N(k) \propto k^D$ active modes in a disk of radius k. The randomness in the choice of the decimation will be called the disorder.

Observe that P_D is a projector, that it commutes with the viscous diffusion operator Λ and that it is self-adjoint for the energy (L^2) norm, defined as usual as $\|\boldsymbol{u}\|^2 \equiv (1/(2\pi)^2) \int |\boldsymbol{u}(x)|^2 d^2 x$, where the integral is over a $2\pi \times 2\pi$ periodicity square. The conservation of energy (by the nonlinear term) for sufficiently smooth solutions of the Navier-Stokes equation can be expressed as $(\boldsymbol{u}, B(\boldsymbol{u}, \boldsymbol{u})) = 0$, where $(\boldsymbol{u}, \boldsymbol{w}) \equiv (1/(2\pi)^2) \int \boldsymbol{u}(x) \cdot \boldsymbol{w}(x) d^2 x$ is the L^2 scalar product.

The *decimated Navier-Stokes equation*, written for an incompressible field $\boldsymbol{v}$, takes the following form

$$\partial_t \boldsymbol{v} = P_D B(\boldsymbol{v}, \boldsymbol{v}) + P_D \boldsymbol{f} + P_D \Lambda \boldsymbol{v}. \quad (6)$$

The initial condition is $\boldsymbol{v}_0 \equiv \boldsymbol{v}(t = 0) = P_D \boldsymbol{u}_0$. Thus, at any later time $P_D \boldsymbol{v} = \boldsymbol{v}$. Energy is again conserved; indeed $(\boldsymbol{v}, P_D B(\boldsymbol{v}, \boldsymbol{v})) = 0$, as is seen by moving the self-adjoint operator P_D to the left hand side of the scalar product and using $P_D \boldsymbol{v} = \boldsymbol{v}$. For enstrophy conservation, take the curl

of (1); the quadratically nonlinear term in the vorticity equation is then $B_{\mathrm{vort}}(\omega, \omega) \equiv -\boldsymbol{u} \cdot \nabla \omega$, where $\boldsymbol{u}$ is expressed in terms of ω by Biot–Savart. The relation $(\omega, B_{\mathrm{vort}}(\omega, \omega)) = 0$ expresses enstrophy conservation. In the decimated case, the proof of enstrophy conservation is identical to that for energy conservation with B replaced by B_{vort}.

If, in addition to decimation, we apply a Galerkin truncation which kills all the modes having wave numbers beyond a threshold K_G, the surviving modes constitute a dynamical system having a finite number of degrees of freedom. Such truncated systems with no forcing and no viscosity have been studied by Lee, Kraichnan and others [8]. Using suitable variables related to the real and imaginary parts of the active modes, the dynamical equations may be written as $\dot{y}_\alpha = \sum_{\beta\gamma} A_{\alpha\beta\gamma} y_\beta y_\gamma$.

For the purely Galerkin-truncated (not decimated) case it is well known that the above dynamical system satisfies a Liouville theorem $\sum_\alpha \partial \dot{y}_\alpha / \partial y_\alpha = 0$ and thus conserves volume in phase space. This in turn implies the existence of (statistically) invariant Gibbs states for which the probability is a Gaussian, proportional to $e^{-(\alpha E + \beta \Omega)}$, where $E = \sum_k |\hat{\boldsymbol{u}}_k|^2$ is the energy and $\Omega = \sum_k k^2 |\hat{\boldsymbol{u}}_k|^2$ is the enstrophy. Such *Gibbs states*, called by Kraichnan *absolute equilibria*, play an important role in his theory of the two-dimensional (2D) inverse energy cascade [9]. If we now combine inviscid, unforced Galerkin truncation and decimation, it is easily checked that the Liouville theorem still holds, provided the decimation preserves Hermitean symmetry. For such Gibbs states, and any active mode ($\theta_k = 1$), one easily checks that the mean square energy $\langle |\boldsymbol{u}_k|^2 \rangle = C'/(\alpha + \beta k^2)$, where $C' > 0$ does not depend on $\boldsymbol{k}$. The corresponding energy spectrum is the mean energy $E(k)$ of modes having a wave number between k and $k + 1$. Up to fluctuations of the disorder, the number of active modes in such a shell is $O(k^{D-1})$. Thus,

$$E(k) = \frac{k^{D-1}}{\alpha + \beta k^2}; \qquad \beta > 0, \qquad \alpha > -\beta, \qquad (7)$$

where various positive constants have been absorbed into a new definition of α and β. An instance is *enstrophy equipartition*: $\alpha = 0$ (all the modes have the same enstrophy), for which the energy spectrum is $E(k) \propto k^{D-3}$. As observed in LPP, this equilibrium spectrum coincides with the Kolmogorov 1941 $k^{-5/3}$ spectrum at the critical dimension $D_c = 4/3$. Note that such Gibbs states are only conditionally Gaussian, for a given disorder. Otherwise, they are highly intermittent, since a given high-k mode will be active only in a small fraction of the disorder realizations. We also note that similar phenomena have been observed in shell models [10].

The form (7) of the D-dimensional absolute equilibria also allows for the kind of Bose condensation in the gravest modes (here, those with unit wave number) found by Kraichnan for 2D turbulence. For this the "inverse

temperature" α must be taken negative, close to its minimum realizable value $-\beta$. The arguments used by Kraichnan to predict an inverse Kolmogorov $k^{-5/3}$ energy cascade for high-Reynolds number 2D turbulence with forcing near an intermediate wave number k_{inj} carry over to the decimated case with $D < 2$. In particular the conservation of enstrophy blocks energy transfer to high wave numbers. This in itself does not imply that the energy will cascade to wave numbers smaller than k_{inj}, producing a k-independent energy flux: it might also linger around and accumulate near k_{inj}.

It is now our purpose to show that for $4/3 < D \leq 2$, when the energy spectrum is prescribed to be $E(k) = k^{-5/3}$ over the inertial range, there is a *negative energy flux* Π_E, vanishing linearly with $D - 4/3$ near the critical dimension $D_c = 4/3$. For this we shall assume that a key feature of the two-dimensional energy cascade carries over to lower dimensions, namely, the existence of scaling solutions with local (in Fourier space) dynamics, so that the energy transfer is dominated by triads of wave numbers with comparable magnitudes. Let us now decompose the energy inertial range into bands of fixed relative width, say one octave, delimited by the wave numbers 2^0, 2^1, 2^2, etc. Because of locality there is much intraband dynamics but, of course, interband interactions are needed to obtain an energy flux. Pure intraband dynamics (with no forcing and dissipation) would lead to thermalization. For dimensional reasons, thermalization and interband transfer have the same time scale, namely, the eddy turnover time $k^{-3/2} E^{-1/2}(k)$.

To get a handle on the combined intraband and interband dynamics we perform a thermodynamic thought experiment in which we artificially separate them in time. In the first phase, starting from a $k^{-5/3}$ spectrum we prevent the various bands from interacting by introducing (impenetrable) interband barriers at their edges. In each band, the modes will then thermalize and achieve a Gibbs state with a spectrum (7) in which α and β are determined by the constraints that the total band energy and enstrophy be the same as for the $-5/3$ spectrum. For example, in the first band this gives the constraints ($n = 0$ for the energy and $n = 2$ for the enstrophy)

$$\int_1^2 dk\, k^n [k^{D-1}/(\alpha + \beta k^2) - k^{-5/3}] = 0, \qquad (8)$$

a system of two transcendental equations for the parameters α and β, which we solved numerically. For $D = 2$, the corresponding absolute equilibrium spectrum, obtained by substituting these values in (7), is shown in Fig. 1, together with the $-5/3$ spectrum. The two spectra are very close to each other. Specifically, in 2D the absolute equilibrium spectrum exceeds the $-5/3$ spectrum by about 10% at any lower band edge and by about 5% at any upper band edge. Of course, as we approach the critical dimension $D_c = 4/3$ the discrepancy goes to zero and can easily be
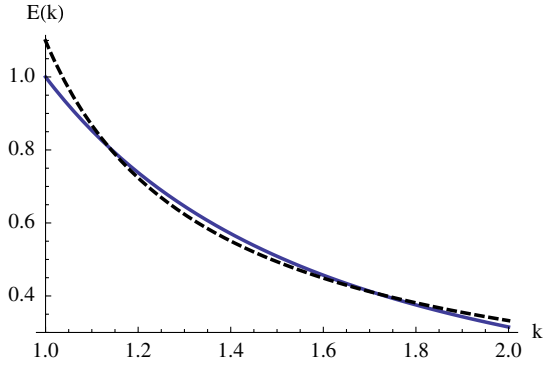
FIG. 1 (color online).   The $k^{-5/3}$ spectrum (continuous) and the associated 2D absolute equilibrium with the same energy and enstrophy in the first octave (dashed).

calculated perturbatively in $D - 4/3$. In the second phase of our thought experiment, we consider two adjacent bands, e.g., $[2^0, 2^1]$ and $[2^1, 2^2]$ that have thermalized, starting from the same $k^{-5/3}$ spectrum and we remove the interband barrier at 2^1. A new thermalization leads then to an absolute equilibrium in the band $[2^0, 2^2]$, which again, can be easily calculated. In 2D, before the removal, the energy between 2^0 and 2^1 was 0.555. After the new thermalization, this energy is found to have *increased* by 0.005 51. Thus energy has been transferred from the upper band $[2^1, 2^2]$ to the lower band $[2^0, 2^1]$. Close to $D_c = 4/3$, we can again apply elementary perturbation techniques and obtain for the upper-to-lower-band energy transfer $0.009(D - 4/3)$ to leading order. Our thermodynamic thought experiment thus suggests that the energy flux vanishes linearly with $D - 4/3$, being negative above the critical dimension, which implies an inverse cascade. Variants of this thought experiment involving more bands give similar results. In the K41 inertial range, the energy spectrum and the energy flux Π_E are related by $E(k) = C_{\mathrm{Kol}} |\Pi_E|^{2/3} k^{-5/3}$, where C_{Kol} is the Kolmogorov constant; thus the Kolmogorov constant diverges as $(D - 4/3)^{-2/3}$. A closure calculation of eddy-damped quasinormal Markovian (EDQNM) type also predicts a divergence with a $-2/3$ exponent.

Kraichnan's ideas about the inverse cascade in 2D got growing support a few years later from direct numerical simulations, which eventually achieved the resolution of $32\,768^2$ modes [11]. As to our idea about the robustness of the inverse cascade and the growth of the Kolmogorov constant when lowering the dimension D, some support can be already provided, using a D-dimensional decimated variant of spectral direct numerical simulation: First one generates an instance of the disorder, that is the list of active and inactive Fourier modes; then, one applies standard time marching algorithms and, at each time step, sets to zero all inactive modes. In addition to the well-known difficulties of simulating 2D turbulence (see, e.g., [11] and references therein), there are new difficulties.

A few words about the numerical implementation. We integrate the decimated Navier-Stokes Eq. (6) in vorticity representation. Instead of using as damping the viscous operator $\Lambda = \nu\Delta$ (where $\Delta \equiv \nabla^2$ is the Laplacian), we use

$$\Lambda \equiv -\nu\Delta^{+2} - \mu\Delta^{-2}, \qquad \nu > 0, \qquad \mu > 0, \quad (9)$$

whose Fourier symbol is $-\nu k^4 - \mu k^{-4}$. In other words, we use hyperviscosity to avoid wasting resolution on the enstrophy cascade and large-scale friction to prevent an accumulation of energy on the gravest modes and thus allow eventual convergence to a statistical steady state. The results reported here have a resolution of $N = 3072$ collocation points in the two coordinates. Time marching is done by an Adams-Bashforth scheme combined with exponential time difference (ETD) [12] with a time step between 5×10^{-5} and 10^{-4}, depending on dimension. Energy injection at the rate ε is done in a band of width three around $k_{\mathrm{inj}} = 319$ by adding to the time rate of change of the Fourier amplitude of the vorticity a term proportional to the inverse of its complex conjugate [13]. This allows a $\boldsymbol{k}$-independent and time-independent energy injection. As D is decreased the amplitude of this forcing is increased to keep the total energy injection on *active* modes fixed at $\varepsilon = 0.01$. The damping parameters are $\nu = 10^{-11}$ and $\mu = 0.1$. Runs are done concurrently for different values of D on a high-performance cluster at the Weizmann Institute and take typically a few thousand hours of CPU per run to achieve a statistical steady state.

Energy spectra are obtained by angular averages over Fourier-space shells of unit width

$$E(K) \equiv \frac{1}{2} \sum_{K \leq k < K+1} |\hat{\boldsymbol{v}}(\boldsymbol{k})|^2, \qquad (10)$$
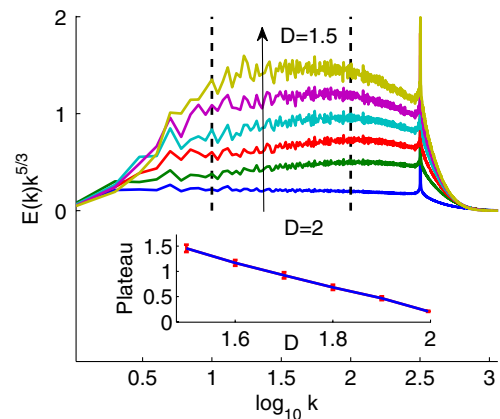

FIG. 2 (color online).   Compensated steady-state spectra for $D = 2.0, 1.9, 1.8, 1.7, 1.6, 1.5$ from bottom to top with spikes at injection. The inset shows the dependence on D of the plateau of the compensated spectra, as an average over the interval between vertical dashed lines (with standard deviation error bars).
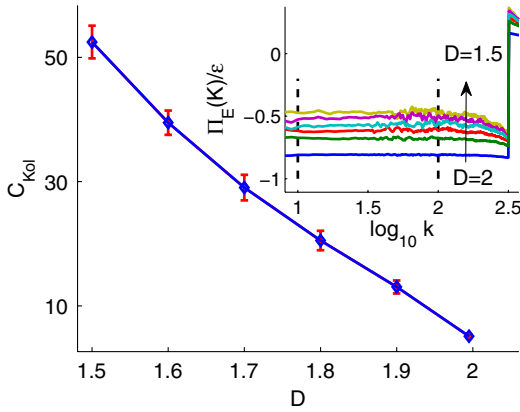
FIG. 3 (color online). Dependence of the Kolmogorov constant on D. The lowest value, at $D = 2$, is about 5. The inset shows the energy flux normalized by the energy injection ε for the same values of D as in Fig. 2.

where the $\hat{\boldsymbol{v}}(\boldsymbol{k})$ are the Fourier coefficients of the solution of the decimated Navier-Stokes Eq. (6). We also need the energy flux $\Pi_E(K)$ through wave number K due to nonlinear transfer, defined as

$$\Pi_E(K) \equiv \sum_{k \leq K} \hat{\boldsymbol{v}}^*(\boldsymbol{k}) \cdot \widehat{\mathrm{NL}}(\boldsymbol{k}), \qquad (11)$$

where $\widehat{\mathrm{NL}}(\boldsymbol{k})$ denotes the set of Fourier coefficients of the nonlinear term $P_D B(\boldsymbol{v}, \boldsymbol{v})$ in the decimated Navier-Stokes Eq. (6) and the asterisk denotes complex conjugation. $E(k)$ and $\Pi_E(K)$ are mostly insensitive to the disorder realization.

Figures 2 and 3 (inset) show the steady-state *compensated* energy spectra $k^{5/3} E(k)$ and the energy fluxes $\Pi_E(k)$, for various values of D between 2 and 1.5, respectively. Both are quite flat, over a significant range of k values, evidence that D-dimensional forced turbulence, Fourier decimated down from the two-dimensional case, preserves the key feature of two-dimensional turbulence of having an inverse cascade that follows the $-5/3$ law. Note that the inertial range (the flat region of the compensated energy spectrum) shrinks as the dimension D decreases. The absolute value of the energy flux is about 80% of the energy injection ε for $D = 2$, but drops to less than 50% for $D = 1.5$. Indeed, as the dimension D is lowered, there are fewer and fewer pairs of *active* modes in the forcing band, capable through their beating interaction of draining the energy into the infrared direction; thus the energy injection will be more and more balanced by direct dissipation near injection. Preventing this would require a substantial lowering of the dissipation which in turn requires a substantial increase in the resolution at the high-k end. Anyway, the fact that the flux $|\Pi_E|$ becomes substantially lower than injection does not prevent us from calculating the Kolmogorov constant, given (in terms of plateau values) by $C_{\mathrm{Kol}} = k^{5/3} E(k)/(|\Pi_E(k)|^{2/3})$. Figure 3 shows the variation of the Kolmogorov constant with dimension.

When lowering the dimension from 2 to 1.5, a combined effect of a rise in the compensated spectrum and a drop in flux yields a monotonic growth of about a factor ten in the Kolmogorov constant and a substantial growth of errors due to fluctuations within the averaging interval. Probing the conjectured divergence by moving closer to the critical point $D_c = 4/3$ would require much higher resolution. A state-of-the-art $16\,384^2$ simulation of sufficient length might shed light.

We finally observe that the fractal Fourier decimation procedure—that allows numerical experimentation by spectral simulation—can be started from any integer dimension and can be applied to a large class of problems in compressible and incompressible hydrodynamics and MHD. It is also applicable to other problems in nonlinear dynamics and condensed matter physics, such as critical dynamical phenomena [14], Kardar–Parisi–Zhang dynamics [15], and nonlinear wave interactions [16].

[1] G. 't Hooft and M. Veltman, Nucl. Phys. B **B44**, 189 (1972).
[2] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
[3] J.-D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1978).
[4] V. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. Lett. **89**, 064501 (2002).
[5] Y. Gefen, A. Aharony, and B. B. Mandelbrot, J. Phys. A **17**, 1277 (1984); B. O'Shaughnessy and I. Procaccia, Phys. Rev. A **32**, 3073 (1985).
[6] The lattice structure that is common to lattice Boltzmann models may be amenable to fractal decimation [7].
[7] H. Chen (private communication).
[8] T. D. Lee, Q. Appl. Math. **10**, 69 (1952); R. H. Kraichnan, J. Acoust. Soc. Am. **27**, 438 (1955).
[9] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[10] E. Aurell, G. Boffetta, A. Crisanti, P. Frick, G. Paladin, and A. Vulpiani, Phys. Rev. E **50**, 4705 (1994); T. Gilbert, V. S. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. Lett. **89**, 074501 (2002).
[11] G. Boffetta and S. Musacchio, Phys. Rev. E **82**, 016307 (2010).
[12] S. M. Cox and P. C. Matthews, J. Comput. Phys. **176**, 430 (2002)
[13] Z. Xiao, M. Wan, S. Chen, and G. Eyink, J. Fluid Mech. **619**, 1 (2009).
[14] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[15] J. P. Bouchaud and M. E. Cates, Phys. Rev. E **47**, R1455 (1993), and references therein.
[16] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992).