Hall Viscosity and Electromagnetic Response

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We show that, for Galilean invariant quantum Hall states, the Hall viscosity appears in the electromagnetic response at finite wave numbers q. In particular, the leading q dependence of the Hall conductivity at small q receives a contribution from the Hall viscosity. The coefficient of the q^2 term in the Hall conductivity is universal in the limit of strong magnetic field.

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Introduction.—Quantum Hall states have been shown to possess, in addition to the Hall conductivity, a new property called the Hall viscosity [1,2]. The Hall viscosity breaks parity, is dissipationless and can be defined at zero temperature. It has been shown recently [3,4] that the Hall viscosity is related to a topological property of the quantum Hall state—the Wen-Zee shift [5].

One may ask how the Hall viscosity can be measured. As originally defined, the Hall viscosity is related to the stress response of the system to metric perturbations. Such perturbations can be, in principle, mimicked by lattice vibrations (sound waves). It has also been suggested that the Hall viscosity determines the stress created by an inhomogeneous electric field [6]. In this Letter we show that, for quantum Hall states of systems with Galilean invariance and made up of particles of the same charge-mass ratio, the Hall viscosity can be, in principle, determined from electromagnetic response alone. We shall show this result first using intuitive physical arguments, and then by employing the formalism of nonrelativistic diffeomorphism invariance, applied to the low-energy effective action of the Hall liquid.

Main result.—Consider a quantum Hall state in finite magnetic field B. First we concentrate on the case when the interaction between particles is short-ranged. (The case of Coulomb interaction will be treated later in the paper.) Let us turn on a static longitudinal electric field $\mathbf{E} = -\nabla\phi$ where ϕ is the scalar potential. We take ϕ to vary in space with some wave-vector $\mathbf{q}$ pointing along the x direction and measure the Hall current j_y (see Fig. 1). The proportionality between j_y and E_x is the wave-vector dependent Hall conductivity,

$$j_y(q) = \sigma_{xy}(q)E_x(q). \tag{1}$$

In the limit $q \to 0$, $\sigma_{xy}(q)$ approaches the universal value, determined by the rational filling factor ν: $\sigma_{xy}(0) = \nu e^2/(2\pi\hbar)$. In general, σ_{xy} has a nontrivial dependence on the wave number q.

We will show that, for a Galilean invariant system of electrons, the coefficient C_2 of the first correction in the low-q expansion of the Hall conductivity

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + C_2(q\ell)^2 + \mathcal{O}(q^4\ell^4), \tag{2}$$

can be related to the Hall viscosity η^a and the function $\epsilon(B)$ which is the energy density (energy per unit area) as function of the external magnetic field $\epsilon(B)$ at fixed filling factor,

$$C_2 = \frac{\eta^a}{\hbar n} - \frac{2\pi}{\nu}\frac{\ell^2}{\hbar\omega_c}B^2\epsilon''(B). \tag{3}$$

Here, $\ell = \sqrt{\hbar c/|e|B}$ is the magnetic length, $\omega_c = |e|B/mc$ is the cyclotron frequency, and n is the density of electrons.

Using the relationship between η^a and the shift $\mathcal{S}$: $\eta^a = \hbar n\mathcal{S}/4$ [3,4], the first term in the right-hand side of Eq. (3) can be written as $\mathcal{S}/4$, which makes clear that the magnitude of this contribution is robust (i.e., does not depend on interactions). The second contribution involves the function $\epsilon(B)$ and is not universal. However, its magnitude can be extracted independently by measuring currents created by weak inhomogeneous perturbations of the magnetic field δB,

$$\mathbf{j} = -c\epsilon''(B)\hat{\mathbf{z}} \times \nabla\delta B. \tag{4}$$

Hence, by measuring the electromagnetic response of the system to inhomogenous electric and magnetic fields, one can determine the Hall viscosity.

The situation becomes simpler in the limit of high magnetic fields (i.e., that of no mixing between Landau levels) in which the energy $\epsilon(B)$ becomes that of noninteracting electrons in a magnetic field. For the integer quantum Hall state with $\nu = N$, the energy density $\epsilon(B) = (N^2/4\pi)\hbar\omega_c/\ell^2$, and the shift $\mathcal{S} = N$, so we have

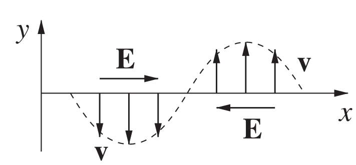

FIG. 1. Pattern of flow in an inhomogeneous electric field.

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 - \frac{3N}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4) \quad \text{for } \nu = N. \tag{5}$$

The result coincides with what has been computed in the literature (σ_{xy} is proportional to Σ_1 in the notation of Ref. [7]). For fractional quantum Hall states with $\nu < 1$, $\epsilon(B) = (\nu/4\pi)\hbar\omega_c/\ell^2$, therefore $C_2 = \frac{1}{4}\mathcal{S} - 1$. In particular, for Laughlin's states with $\nu = 1/(2k+1)$, the shift $\mathcal{S} = 2k+1$ [5], so

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + \frac{2k-3}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4), \quad \nu = \frac{1}{2k+1}. \tag{6}$$

In general, for any quantum Hall state, we can find the q^2 correction to $\sigma_{xy}(q)$ from the value of the shift $\mathcal{S}$ and the total energy, as a function of the magnetic field.

Physical argument.—Before presenting the mathematical proof of the statement made above, we will give a physical derivation. We will show that the two contributions to C_2 come from two different physical effects.

First let us note that to first approximation, the Hall fluid moves along the y direction with a velocity that depends on x (see Fig. 1),

$$v_y(x) = -\frac{cE_x(x)}{B}. \tag{7}$$

This velocity is determined by balancing electric and magnetic forces acting on a fluid volume. However, the flow (7) is a shear flow with a nonzero strain rate. The Hall viscosity leads to an additional stress in the system, which in turn induces a correction to the current.

Let us compute the magnitude of the this effect. The strain rate $V_{xy} = \frac{1}{2}\partial_x v_y$ induces, through the Hall viscosity, an additional contribution to the stress, $\tau_{xx} = -\tau_{yy} = 2\eta^a V_{xy}$ The x dependence of τ_{xx} leads to an additional force acting on each volume element of the fluid along the x axis: $f_x = -\partial_x \tau_{xx}$. This force induces a correction to the Hall current equal to

$$\delta j_y = -\frac{c}{B} f_x = -\frac{\eta^a c^2}{B^2} E_x''(x). \tag{8}$$

We thus find the first correction to σ_{xy},

$$\sigma_{xy}^{(1)}(q) = \frac{\eta^a c^2}{B^2} q^2. \tag{9}$$

The second effect is related to the fact that the fluid flow, in addition to having a shear rate, also has a nonzero local angular velocity:

$$\Omega(x) = \frac{1}{2}\partial_x v_y = -\frac{cE_x'(x)}{2B}. \tag{10}$$

This local rotation acts as an effective magnetic field, equal to $\delta B = 2mc\Omega/e$ (found by equating the Coriolis force with the Lorentz force from δB.) On the other hand, the quantum Hall fluid is a diamagnetic material. with magnetic moment density $M = -\partial\epsilon/\partial B$. For a constant magnetic field, M is constant. But due to the fluctuations δB there is an inhomogeneous contribution to the magnetic moment density,

$$\delta M = -\frac{\partial^2\epsilon}{\partial B^2}\delta B = \epsilon''(B)\frac{mc^2 E_x'(x)}{eB}. \tag{11}$$

This fluctuating magnetic moment density leads to an additional electromagnetic current, $\mathbf{j} = c\hat{\mathbf{z}} \times \nabla M$:

$$j_y = \epsilon''(B)\frac{mc^3 E_x''(x)}{|e|B}. \tag{12}$$

We find the second contribution to the Hall conductivity,

$$\sigma_{xy}^{(2)}(q) = -\frac{mc^3\epsilon''(B)}{|e|B} q^2. \tag{13}$$

The finite-wave-number correction to the Hall conductivity is $\sigma_{xy}^{(1)} + \sigma_{xy}^{(2)}$. Elementary algebraic manipulations bring it to the form of Eqs. (2) and (3).

Diffeomorphism invariance.—We now formally prove the result derived above by constructing a low-energy effective theory of the quantum Hall state. As the quantum Hall state is gapped, the effective action is as a local functional of the external fields. Expanding in momentum to lowest order, it is simply the Chern-Simons action. In order to reproduce the q^2 correction to σ_{xy} we need to go beyond leading order.

We shall make use of the nonrelativistic diffeomorphism invariance, introduced in Ref. [8]. Our strategy is to couple our system to gravity and find out the symmetries of the action. These symmetries are inherited by the low-energy effective theory, and impose nontrivial constraints to the effective Lagrangian.

We consider a quantum Hall state in the presence of an external gauge field $A_\mu(t,\mathbf{x})$ and a spatial metric $g_{ij}(t,\mathbf{x})$. For example, for the case of free fermions we assume the action to be

$$S_0 = \int dt d^2x\sqrt{g}\Big[\frac{i}{2}(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi) + A_0\psi^\dagger\psi - \frac{g^{ij}}{2m}(\partial_i\psi^\dagger + iA_i\psi^\dagger)(\partial_j\psi - iA_j\psi)\Big]. \tag{14}$$

We will set $\hbar = 1$ and absorb an e/c factor into the normalization of the gauge potential A_i. Most of the time we will set the spatial metric to be flat ($g_{ij} = \delta_{ij}$) at the end of calculations, but it will be useful to consider a general metric in the intermediate steps.

The action (14) is invariant under reparametrization of spatial coordinates $x^k \to x^k + \xi^k$, where ξ^k depends both on space and time, $\xi^k = \xi^k(t,\mathbf{x})$. The passive form of the transformations is

$$\delta A_0 = -\xi^k\partial_k A_0 - A_k\dot{\xi}^k, \tag{15}$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k - m g_{ik} \dot{\xi}^k, \quad (16)$$

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{kj} \partial_i \xi^k - g_{ik} \partial_j \xi^k, \quad (17)$$

$$\delta \psi = -\xi^k \partial_k \psi. \quad (18)$$

The Galilean transformation is a special case with $\xi^k = v^k t$. As explained in Ref. [8], the transformations above can be motivated by taking a nonrelativistic limit of relativistic diffeomorphisms.

Interactions can be introduced in a way which preserves the diffeomorphism invariance. For example, by adding to (14)

$$S = S_0 + \int dt d^2x \sqrt{g} \left(\lambda \psi^\dagger \psi \phi - \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi - \frac{m_\phi^2}{2} \phi^2 \right) \quad (19)$$

one introduces an attractive potential of range m_ϕ^{-1} between the particles. The new action is diffeomorphism invariant if ϕ transforms as a scalar $\delta\phi = -\xi^k \partial_k \phi$. A generic potential decaying faster than an exponential can be represented by an infinite number of mediating fields, and so coupling to the external metric can be made compatible with diffeomorphism invariance.

Coulomb interactions can also be introduced, but now the field mediating the interaction propagates in three spatial dimensions. We can assume that the spatial metric does not depend on the third direction

$$S = S_0 + \int dt d^2x \sqrt{g} a_0 (\psi^\dagger \psi - n_0) + \frac{2\pi\varepsilon}{e^2} \int dt d^2x dz \sqrt{g} [g^{ij} \partial_i a_0 \partial_j a_0 + (\partial_z a_0)^2]. \quad (20)$$

(ε is the dielectric constant). We have included a neutralizing background with density n_0. The full action is diffeomorphism invariant if a_0 transforms as a scalar: $\delta a_0 = -\xi^k \partial_k a_0$.

Power counting.—We now start constructing the low-energy effective field theory of the quantum Hall states. For incompressible states, there is no low-energy excitations, and we can integrate out ψ. If interactions are short-ranged, the fields ϕ mediating interactions can also be integrated out. Thus the effective Lagrangian is a local function of the external fields A_μ, g_{ij} and their derivatives. The effective action must be invariant under (15)–(17).

To organize a derivative expansion, one needs a power-counting scheme with a small parameter. There is an ambiguity in choosing the scheme, as the time derivative ∂_t and spatial derivatives can be chosen to be independent expansion parameters. For definiteness, in this Letter we use the following scheme. All quantities will be regarded as proportional to some powers of a small parameter ϵ, times some powers of ω_c and ℓ. The external fields are assumed to vary slowly in space and time,

$$\partial_i \sim \epsilon \ell^{-1}, \qquad \partial_t \sim \epsilon^2 \omega_c. \quad (21)$$

As for the magnitude of external perturbations, we assume

$$\delta A_0 \sim \epsilon^0 \omega_c, \qquad \delta A_i \sim \epsilon^{-1} \ell^{-1}, \qquad \delta g_{ij} \sim 1. \quad (22)$$

In this scheme, we allow for order one variations of the metric, the magnetic field ($\delta B \sim \epsilon^0 \ell^{-2}$) and the chemical potential (A_0). In further formulas, the electric and magnetic fields are defined as

$$E_i = \partial_i A_0 - \partial_0 A_i, \quad B = \frac{F_{12}}{\sqrt{g}} = \frac{\epsilon^{ij} \partial_i A_j}{\sqrt{g}} \equiv \varepsilon^{ij} \partial_i A_j, \quad (23)$$

so $E_i = O(\epsilon)$ and $B = O(1)$.

Chern-Simons and Wen-Zee terms.—Two important ingredients in our construction of the effective field theory are the Chern-Simons action and the Wen-Zee action. The Chern-Simons action is

$$S_{\rm CS} = \frac{\nu}{4\pi} \int dt d^2x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \quad (24)$$

and is of order ϵ^0 in our power-counting scheme. This will be the leading term in the effective action. To construct the Wen-Zee action, we first define the spin connection. We introduce a spatial vielbein e_i^a, $a = 1, 2$ so that $g_{ij} = e_i^a e_j^a$ and $\epsilon^{ab} e_i^a e_j^b = \varepsilon_{ij}$. The vielbein is defined up to local $O(2)$ rotations in a space. If we now define the connection ω_μ,

$$\omega_0 = \tfrac{1}{2} \epsilon^{ab} e^{aj} \partial_0 e_j^b, \quad (25)$$

$$\omega_i = \tfrac{1}{2} \epsilon^{ab} e^{aj} \nabla_i e_j^b = \tfrac{1}{2} (\epsilon^{ab} e^{aj} \partial_i e_j^b - \varepsilon^{jk} \partial_j g_{ik}), \quad (26)$$

then under local $O(2)$ rotations ω_μ transforms like an Abelian gauge potential $\omega_\mu \to \omega_\mu - \partial_\mu \lambda$. By using ω_μ we can construct the following gauge invariant action

$$S_{\rm WZ} = \frac{\kappa}{2\pi} \int dt d^2x \epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu A_\lambda. \quad (27)$$

This action is of order ϵ^2 in our power-counting scheme and has to be included if we work to that order. The $\omega d\omega$ Chern-Simons term, on the other hand, is of order ϵ^4 and will not be considered.

The coefficient κ is related to the shift $\mathcal{S}$. Indeed, the "torsion magnetic" field $\partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2} \sqrt{g} R$ where R is the scalar curvature. Integrating by parts, the Wen-Zee action contains a term

$$\frac{\kappa}{2\pi} \epsilon^{\mu\nu\lambda} \omega_\mu \partial_\nu A_\lambda \simeq \frac{\kappa}{4\pi} \sqrt{g} A_0 R + \cdots, \quad (28)$$

which gives a contribution to the particle number density that is proportional to the scalar curvature. If the quantum Hall state lives on a closed two dimensional surface, then the total number of particles is

$$Q = \int d^2x\sqrt{g}j^0 = \int d^2x\sqrt{g}\left(\frac{\nu}{2\pi}B + \frac{\kappa}{4\pi}R\right) = \nu N_\phi + \kappa\chi, \tag{29}$$

where N_ϕ is the total number of magnetic fluxes and $\chi = 2(1-g)$ is the Euler character. Comparing to the definition of $\mathcal{S}$ in Ref. [5], we find $\kappa = \frac{1}{2}\nu\mathcal{S}$. For the integer Quantum Hall state with $\nu = N$, $\kappa = N^2/2$. For Laughlin's states $\kappa = 1/2$.

The Wen-Zee action gives rise to Hall viscosity [9]. Expanding the WZ term to quadratic order, one finds, among other terms,

$$S_{\rm WZ} = -\frac{\kappa B}{16\pi}\epsilon^{ij}\delta g_{ik}\partial_t\delta g_{jk} + \cdots, \tag{30}$$

which implies the presence of an odd term in the stress tensor two-point function, or Hall viscosity. The value of the Hall viscosity is $\eta^a = \kappa B/4\pi = \frac{1}{4}\mathcal{S}n$. This relationship between the Hall viscosity and the shift has been derived previously in Ref. [4].

Most general effective action.—It is straightforward to verify that both $S_{\rm CS}$ and $S_{\rm WZ}$ are not diffeomorphism invariant, and need to be corrected. In fact, to order $O(\epsilon^2)$, the most general effective action can be written as $S = \int dt d^2x\sqrt{g}\sum_{i=1}^5 \mathcal{L}_i$, where $\mathcal{L}_i$ $(i = 1,\ldots,5)$ are five independent general diffeomorphism invariant (to order ϵ^2) terms

$$\mathcal{L}_1 = \frac{\nu}{4\pi}\left(\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{m}{B}g^{ij}E_iE_j\right), \tag{31}$$

$$\mathcal{L}_2 = \frac{\kappa}{2\pi}\left(\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu A_\lambda + \frac{1}{2B}g^{ij}\partial_i BE_j\right), \tag{32}$$

$$\mathcal{L}_3 = -\epsilon(B) - \frac{m}{B}\epsilon''(B)g^{ij}\partial_i BE_j, \tag{33}$$

$$\mathcal{L}_4 = -\tfrac{1}{2}K(B)g^{ij}\partial_i B\partial_j B, \tag{34}$$

$$\mathcal{L}_5 = Rh(B), \tag{35}$$

where $\epsilon(B)$, $K(B)$, and $h(B)$ are functions of B. The function $\epsilon(B)$ has the physical meaning of the energy density of the quantum Hall state as a function of the magnetic field B, $\mathcal{L}_4$, and $\mathcal{L}_5$ do not enter the quantities of our interest. The next to leading order term in $\mathcal{L}_1$ enforces compliance with Kohn's theorem. The two-point function of the electromagnetic current j^μ is obtained by taking the second derivative of the effective action with respect to A_μ, then setting perturbations to zero. Equivalently, we can differentiate the effective action once with respect to the external fields to get the current. We find in flat space

$$j^i = \frac{\nu}{2\pi}\epsilon^{ij}E_j - \frac{1}{B}\left[\frac{\kappa}{4\pi} - m\epsilon''(B)\right]\epsilon^{ij}\partial_j(\nabla\cdot\mathbf{E}) + \cdots, \tag{36}$$

where ... refers to term that vanish when the magnetic field is not perturbed. Equations (2) and (3) are reproduced from this formula.

Inclusion of Coulomb interactions.—In the case with Coulomb interactions, one needs to take into account the screening of the electric field. The expansion (2) and (3) therefore applies not to $\sigma_{xy}(q)$ but to

$$\tilde{\sigma}_{xy}(q) = \left[1 + \frac{e^2\chi(q)}{2\pi\epsilon q}\right]\sigma_{xy}(q) \simeq \left[1 + \frac{\nu\aleph}{\pi}(q\ell)\right]\sigma_{xy}(q), \tag{37}$$

where $\aleph = e^2/(4\pi\epsilon\ell\omega_c)$ and $\chi(q)$ is the static susceptibility, the small-q behavior of which is determined by Kohn's theorem: $\chi(q) = \nu mq^2/(2\pi B)$. In the limit of high magnetic fields where $\aleph \ll 1$, the distinction between σ_{xy} and $\tilde{\sigma}_{xy}$ disappears.

Conclusions.—We have shown that the Hall viscosity does not only appear in the response to gravitational fluctuations, but also, under certain circumstances, in a purely electromagnetic response function. For this one needs Galilean invariance and that all particles have the same charge/mass ratio, a condition satisfied in the most interesting physical systems.

One notes that topological arguments alone are insufficient to determine the coefficient of the q^2 term in the finite-wave-number Hall conductivity. But topology, coupled with nonrelativistic diffeomorphism invariance, is powerful enough to find this coefficient [e.g., Eq. (6)]. It would be interesting to explore consequences of diffeomorphism invariance for other systems with topological order, e.g., the $p_x + ip_y$ paired state or the superfluid B phase of ^{3}He or the compressible $\nu = 1/2$ state.

Finally, the wave number dependence of the Hall conductivity should be measured and checked against our prediction. Such a measurement would be a measurement of the Hall viscosity.

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Note added.—After this work was finished, we learned that the first contribution on the right-hand side of Eq. (3) has been derived by B. Bradlyn, M. Goldstein, and N. Read [10]. We also learned from I. V. Tokatly that Eq. (3) can be recovered within the model proposed in Ref. [11]. We thank N. Read and I. V. Tokatly for communicating these results to us.

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