Destruction of Adiabatic Invariance for Billiards in a Strong Nonuniform Magnetic Field

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We study a classical billiard of charged particles in a strong nonuniform magnetic field. We provide an adiabatic description for skipping motion along the boundary of the billiard. We show that a sequence of many changes of regimes of motion from skipping to motion without collisions with the boundary and back to skipping leads to destruction of the adiabatic invariance and chaotic dynamics in a large domain in the phase space. This is a new mechanism of the origin of chaotic dynamics for systems with impacts.

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Billiard in a magnetic field is a popular model in nonlinear dynamics with possible applications in theory of magnetism, plasma physics, solid state physics, and astrophysics. In this model the motion of a charged particle in a plane region with a perfectly reflected smooth boundary is considered. A magnetic field is directed perpendicular to the plane. If the magnetic field is strong enough, then skipping along the boundary is possible. Figures 1 (a,b) demonstrate two types of particle trajectories in such a billiard for a uniform magnetic field.

This model was first studied in [1] and then in a series of other papers, in particular, in [2–5]. It is shown in [1], that for a strong uniform magnetic field the distance of the center of the Larmor circle of the particle from the boundary is an adiabatic invariant. The first and the second order corrections to this adiabatic invariant are calculated as well [1,2].

We consider the case of a nonuniform strong magnetic field. We demonstrate that for the skipping motion the flux of the magnetic field through the area bounded by the arc of particle's trajectory between two collisions with the boundary and the corresponding segment of the boundary is an adiabatic invariant (this is an analog of the magnetic moment). We describe the skipping along the boundary in the adiabatic approximation.

There is a very important difference between billiards with uniform and nonuniform magnetic fields. In the latter case the particle can change the regime of motion with skipping along the billiard boundary to the regime of the drift without collisions with the boundary, and vice versa. We show that a sequence of such changes of regime of motion leads to destruction of the adiabatic invariance and chaotic dynamics in a large domain in the phase space. This resembles destruction of adiabatic invariance due to separatrix crossings in slow-fast Hamiltonian systems [6–9].

Our approach to the description of skipping motion is completely different from that in [1–5]. Unlike these papers, we do not use a billiard map and do not consider imagined motion beyond the billiard's boundary. Instead, we consider the billiard as a slow-fast Hamiltonian system and apply an adiabatic perturbation theory for analysis of this system.

The motion of a charged particle in a plane under the action of a magnetic field perpendicular to this plane is described by the Hamiltonian system with the phase variables x, y, p_x, p_y, Hamiltonian H and symplectic structure ω^2 (see, e.g., [10]):

$$\begin{aligned} H &= \frac{1}{2m}(p_x^2 + p_y^2) \\ \omega^2 &= dp_x \wedge dx + dp_y \wedge dy + \varepsilon^{-1}\frac{e}{c}B(x,y)dx \wedge dy \end{aligned} \qquad (1)$$

Here x, y, p_x, p_y are coordinates and momenta of the particle in some Cartesian coordinate system Oxy, m and e are mass and charge of the particle, c is the speed of light, $\varepsilon^{-1}B(x, y)$ is the strength of the magnetic field, ε is a

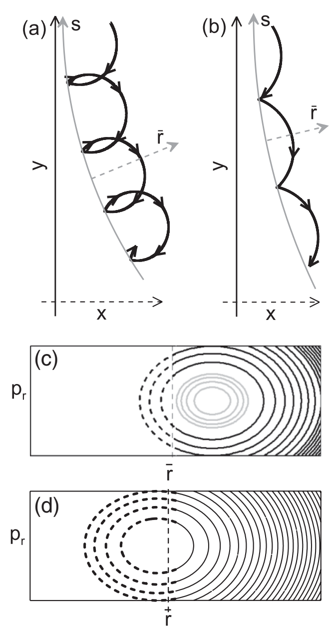

FIG. 1. Two types of trajectories and corresponding phase portraits.

dimensionless positive constant. We consider the case of a strong magnetic field: $\varepsilon \ll 1$, and $B \sim 1$ does not vanish anywhere.

We suppose that in the plane of the motion there is a smooth curve $\mathcal{L}$ (the billiard boundary), and a collision of the particle with $\mathcal{L}$ leads to the perfect (specular) reflection. In what follows $\mathcal{L}$ may be open or close, but it should not have self intersections. The curve $\mathcal{L}$ has a parametric representation $x = X(s)$, $y = Y(s)$, where s is the arclength along $\mathcal{L}$. As it is a standard in billiard problems (see, e.g., [4,5,11]) introduce near $\mathcal{L}$ new coordinates r, s, where r is the distance from $\mathcal{L}$ and s is the arclength of the projection of particle position onto $\mathcal{L}$. Thus $x = X(s) + rY'(s)$, $y = Y(s) - rX'(s)$. Here ($'$) denotes the derivative with respect to s. We will consider the dynamics in the domain $r > 0$.

Introduce new momenta p_r, p_s by means of the canonical transformation $(x, y, p_x, p_y) \mapsto (r, s, p_r, p_s)$ with the generating function $W = xp_x + yp_y$, where x, y are expressed via r, s. Then $p_r = (\partial x/\partial r)p_x + (\partial y/\partial r)p_y$, $p_s = (\partial x/\partial s)p_x + (\partial y/\partial s)p_y$, and $dp_x \wedge dx + dp_y \wedge dy = dp_r \wedge dr + dp_s \wedge ds$. Also $dx \wedge dy = \det(\partial(x, y)/\partial(r, s))\, dr \wedge ds = (1 + k(s)r)dr \wedge ds$, where $k(s) = -Y'X'' + X'Y''$ is the curvature of $\mathcal{L}$. In the new variables the dynamics is described by the following Hamiltonian and symplectic structure:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_s^2}{(1 + k(s)r)^2}\right)$$

$$\omega^2 = dp_r \wedge dr + dp_s \wedge ds + \varepsilon^{-1}(1 + k(s)r)\frac{e}{c}B(r, s)dr \wedge ds$$

(here we keep notation B for the strength of the magnetic field expressed via the new coordinates r, s). Introduce a canonical momentum $\mathcal{P}_s = p_s + \varepsilon^{-1}A(r, s)$, where

$$A(r, s) = \int_0^r (1 + k(s)\xi)\frac{e}{c}B(\xi, s)d\xi = \frac{e}{c}(B_0(s)r + O(r^2)),$$

and $B_0(s) = B(0, s)$. In the new variables the symplectic structure takes the canonical form: $\omega^2 = dp_r \wedge dr + d\mathcal{P}_s \wedge ds$. Thus the dynamics in the new variables is described by the canonical Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2m}\left(p_r^2 + \left(\frac{\mathcal{P}_s - \varepsilon^{-1}A(r, s)}{1 + k(s)r}\right)^2\right) \tag{2}$$

For the description of the dynamics near $\mathcal{L}$ introduce new variables and time: $\bar{r} = r/\varepsilon$, $\bar{s} = s/\varepsilon$, $\bar{t} = t/\varepsilon$. In the new variables the motion between collisions with $\mathcal{L}$ is described by the canonical Hamiltonian system with the Hamiltonian

$$H = H_0(\bar{r}, p_r, \varepsilon\bar{s}, \mathcal{P}_s) + O(\varepsilon\bar{r}^2)$$

where

$$H_0 = \frac{1}{2m}\left[p_r^2 + \left(\mathcal{P}_s - \frac{e}{c}B_0(\varepsilon\bar{s})\bar{r}\right)^2\right].$$

At collisions ($r = 0$), $\bar{s}$ and $\mathcal{P}_s$ are continuous, while p_r changes the sign.

Consider dynamics described by the Hamiltonian H for $\bar{r} \sim 1$. In this case p_r, $\bar{r}$ are fast variables, while $\mathcal{P}_s$, $\varepsilon\bar{s}$ are slow variables. The system under consideration is a slow-fast Hamiltonian system with impacts. For description of dynamics in this system one can use an adiabatic perturbation theory (see, e.g., [10]). This theory was developed for smooth slow-fast Hamiltonian systems, but it can be used for systems with impacts as well [12,13].

Consider the system with Hamiltonian H_0 at frozen values of $s = \varepsilon\bar{s}$ and $\mathcal{P}_s$ (the unperturbed system). This is a Hamiltonian system with 1 degree of freedom: an oscillator with impacts. Phase portraits of this system are shown in Figs. 1(c) and 1(d). Figure 1(c) demonstrates trajectories without collision (shown in the gray color) and with collisions with the boundary. In Fig. 1(d) each trajectory contains a collision. A trajectory with $H_0 = h$ contains a collision if $h > \mathcal{P}_s^2/(2m)$. The action of this trajectory $I(h, s, \mathcal{P}_s)$ is the area surrounded by this trajectory and by the corresponding segment of the p_r axis, divided by 2π. Calculating this area, we get

$$2\pi I = \frac{2hmc}{eB_0(s)}[\pi - \arccos(1 - \delta) + (1 - \delta)\sqrt{\delta(2 - \delta)}] \tag{3}$$

where $\delta = 1 - \mathcal{P}_s/\sqrt{2hm}$. In the exact system, where s and $\mathcal{P}_s$ are changing, this action is an adiabatic invariant: its value along trajectory is conserved with an accuracy $\sim\varepsilon$ over time intervals $\bar{t} \sim 1/\varepsilon$ (thus $t \sim 1$) provided p_r is separated from 0 at collisions (or, which is the same, $\delta > \text{const} > 0$). Inverting Eq. (3) we get $h = h(I, \varepsilon\bar{s}, \mathcal{P}_s)$. The approximation in which $I = \text{const}$ and dynamics of $\bar{s}$, $\mathcal{P}_s$ is described by the canonical Hamiltonian system with Hamiltonian h, is called an adiabatic approximation. In this approximation

$$\frac{ds}{dt} = \frac{\partial h}{\partial \mathcal{P}_s} = -\frac{\partial I/\partial \mathcal{P}_s}{\partial I/\partial h}, \qquad \frac{d\mathcal{P}_s}{dt} = -\varepsilon\frac{\partial h}{\partial s} = \varepsilon\frac{\partial I/\partial s}{\partial I/\partial h}.$$

These equations describe changing of $s = \varepsilon\bar{s}$ and $\mathcal{P}_s$ in skipping motion with accuracy $\sim\varepsilon$. It is easy to check that $I = e\Phi/(2\pi c\varepsilon)$, where $\Phi \sim \varepsilon$ is the flux of the magnetic field through the circular segment bounded by the arc of the particle Larmor trajectory and $\mathcal{L}$. Thus this flux is an adiabatic invariant (value Φ/ε along trajectory is conserved with an accuracy $\sim\varepsilon$ over time intervals $\bar{t} \sim 1/\varepsilon$).

Consider the skipping with initial values of energy h_0 and action I_0. In adiabatic approximation along trajectory $I(h_0, s, \mathcal{P}_s) = I_0$. If along trajectory $B_0(s)$ grows and for some $s = s_*$ we have $mch_0/(eB_0(s_*)) = I_0$, then at $s = s_*$ the Larmor circle recedes from the boundary. After the take off particle moves in accordance with the guiding center theory: the center of its Larmor circle drifts along the line of constant of $B(x, y)$ (see Fig. 2). The speed of this drift $\sim\varepsilon$.

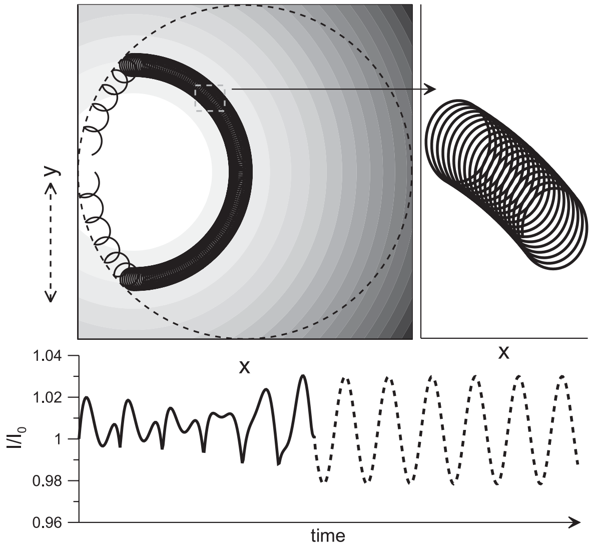

FIG. 2. Particle trajectory in the system with the inhomogeneous magnetic field (strength of the field is shown by color), $B(x, y) = 1 + ((x - 0.5)^2 + y^2)$, $\varepsilon = 0.1$. Bottom panel demonstrates time behavior of the adiabatic invariant I: black curve corresponds to the time interval shown by dotted arrow on the top panel and dotted curve corresponds to the first five Larmor turns after take off.

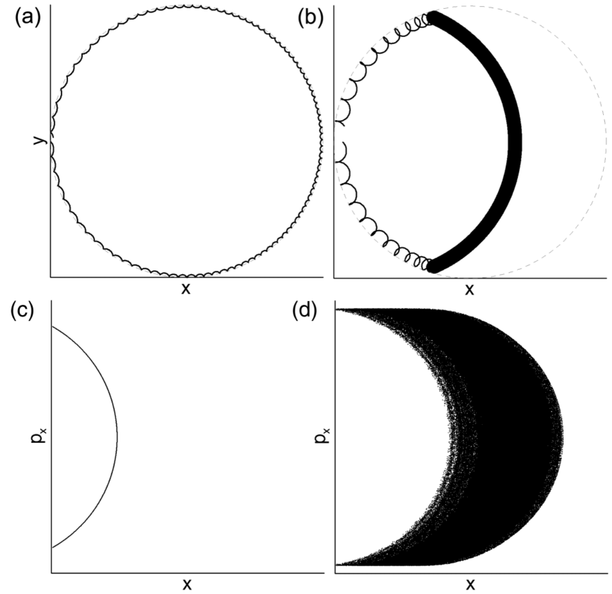

FIG. 3. Fragments of two trajectories (without and with take off) in the system with inhomogeneous magnetic field $B(x, y) = 1 + ((x - 0.5)^2 + y^2)$, $\varepsilon = 0.1$, and corresponding Poincaré sections.

Let $\mathcal{L}$ be a closed curve, and $\max B_0(s) < mch_0/(eI_0)$. Then in the adiabatic approximation there is no take off from the boundary for a particle with $h = h_0$, $I = I_0$. One can show using KAM theory that in this case the value of I is conserved eternally with an accuracy $\sim\varepsilon$, and, in particular, there is no take off in the exact system. This behavior is shown in Figs. 3(a) and 3(c) for the case when $\mathcal{L}$ is the circle $(x - 1.5)^2 + y^2 = 1.5^2$. Figure 3(a) shows a segment of a trajectory of the particle, and Fig. 3(c) shows the Poincaré section at $y = 0$, $p_y < 0$. An invariant curve is clearly seen in Fig. 3(c). Existence of such invariant curves guarantees eternal conservation of the adiabatic invariance of I. For the case of uniform magnetic field ($B = \text{const}$) an eternal adiabatic invariance in a skipping motion is pointed out in [2].

Now consider the case when multiple changes of regimes of the motion occur. The regime of skipping along the boundary changes to a regime of drift in nonuniform magnetic field without collisions with the boundary, which changes to regime of skipping again, and so on. In the adiabatic approximation this dynamics is periodic. The points of landing on the boundary and take off from the boundary are two points where the magnetic field has the same value $\varepsilon^{-1}mch_0/(eI_0)$, and on the arc of the boundary joining these points the magnetic field is smaller than this value. The drift without collisions is described by the guiding center theory, see, e.g., [14]. The guiding center moves along level lines of the function $B(x, y)$ joining points of take off and landing: $B(x, y) = mch_0/(eI_0)$. This motion is described by the Hamiltonian system with the Hamiltonian $E(x, y) = \varepsilon IeB(x, y)/(mc)$, where x and y play roles of a momentum and a coordinate, respectively, $I = I_0 = \text{const}$. The time of skipping from the landing to the take off ~ 1. The time of drift from the take off to the landing $\sim 1/\varepsilon$. (Note that the Larmor period $\sim\varepsilon$).

However, change of regime of motion leads to a jump of the value of adiabatic invariant, cf. [13]. This jump occurs because of nonanalytic dependence of I on δ near take off and landing moments: $2\pi IeB_0(s)/(hmc) - 1 \sim -5\delta^{3/2}/4$. Similarly to [13], one can derive an asymptotic formula for this jump. Here we just estimate amplitude of this jump as follows. One can introduce an improved adiabatic invariant $J = I + \varepsilon u$ (see, e.g., [10]) such that far from the moment of take off the difference of its values at two subsequent collisions with the boundary $\sim\varepsilon^3$. Here u is a function of phase variables, a first correction to the adiabatic invariant. Close to the moment of the take off one should take into account the smallness of δ: using standard formulas of adiabatic perturbation theory (see, e.g., [10]) one can show that the difference of values of J at two subsequent collisions $\sim\varepsilon^3\partial^3 I/\partial\delta^3 \sim \varepsilon^3/\delta^{3/2}$. Here δ is taken at the moment of the first of these two collisions. The decay of δ between two collisions near take off $\sim\varepsilon$. Let us sum up changes of J between $\sim 1/\varepsilon$ collisions before take off and take into account that for the last term in this sum $\delta = \delta_* \sim \varepsilon$. We get that the total change of J between all collisions $\sim\varepsilon^2/\delta_*^{1/2} \sim \varepsilon^{3/2}$. On the first Larmor round after the last collision there is an additional change of J due to difference of its definition for regimes with and without collisions. This change is of order of the area bounded by a dotted line in Fig. 1(c) for $\delta \sim \varepsilon$, which is again $\sim\varepsilon^{3/2}$. Thus the total change of J due to one change of regime of motion $\sim\varepsilon^{3/2}$. This change depends on the value δ_*/ε and can be considered as a function of the phase of the particle on the Larmor circle at the moment of time when this circle touches the boundary at take off (and similarly for landing). This change of J leads to a change of the coefficient I in the Hamiltonian E of the guiding center motion by a value $\sim\varepsilon^{3/2}$. As result, the time of this motion changes by a value $\sim\varepsilon^{3/2}\varepsilon^{-1} = \varepsilon^{1/2}$. This leads to change of Larmor phase $\sim\varepsilon^{1/2}\varepsilon^{-1} = \varepsilon^{-1/2}$ (we use here that Larmor frequency $\sim\varepsilon^{-1}$). This means stretching of the phase and loss of predictability: small change of Larmor phase before take off is stretched with a coefficient $\sim\varepsilon^{-1/2} \gg 1$ at the next take off. The dynamics becomes chaotic. Small changes of the adiabatic invariant for many changes of regime of motion can be considered as a sequence of independent random values. Summation of these values leads to destruction of adiabatic invariance on long time intervals. This is a new mechanism of the origin of chaotic dynamics for systems with impacts. It should be noted that for small enough ε the size of the domain of chaotic motion is of order 1 and does not depend on ε.

This mechanism of the origin of chaotic dynamics is illustrated in Figs. 3(b) and 3(d) for the case when the boundary $\mathcal{L}$ is the circle $(x - 1.5)^2 + y^2 = 1.5^2$. Figure 3(b) shows a segment of a trajectory of the particle, and Fig. 3(d) shows the Poincaré section at $y = 0$, $p_y < 0$.

We observe here a situation usually referred to as a Hamiltonian intermittency (see, e.g., [15]). The motion over a considerably long time occurs along a seemingly regular orbit, but after a passing through a narrow domain in the phase space it randomly switches to another regular orbit. There are many systems that demonstrate such an intermittent behavior with certain underlying mechanisms of switching between regular trajectories (see, e.g., [6–9,16]). Jump of an adiabatic invariant at change of regime of motion is a new such mechanism for systems with impacts.

The dynamics of billiards in the magnetic field can be considered as a simplified (but still realistic) model of the charged particles motion in the vicinity of a strong gradient of the magnetic field. In this case the so-called gradient drift occurs and particles move along the boundary which separates regions with strong and relatively weak magnetic fields [14]. Such a situation is often observed in laboratory plasma [17] and in a space plasma environment, where magnetic field configurations with the strong gradients of the magnetic field and closed field lines (so-called plasmoids and magnetic islands [18], or other magnetic boundaries, e.g., magnetopauses of planetary magnetospheres [19]) exist.

In conclusion, we described a new mechanism responsible for the destruction of adiabatic invariance and the origin of chaotic dynamics in the system with impacts: the sequence of changes of motion types (with and without collisions).

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