Maximal-Helicity-Violating n-Point One-Loop Amplitude in $\mathcal{N} = 4$ Supergravity

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We propose an explicit formula for the n-point maximal-helicity-violating one-loop amplitude in a $\mathcal{N} = 4$ supergravity theory. This formula is derived from the soft and collinear factorizations of the amplitude.

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Introduction.—''Maximal-Helicity-Violating,'' or MHV, amplitudes are scattering amplitudes where exactly two outgoing massless particles have negative helicity and the remaining legs have positive helicity. These objects have been key in many of the recent developments in perturbative gauge theories. The Parke-Taylor formulae [1] gave simple explicit formulae for the MHV tree-level scattering of n gluons in a color-ordered formalism. The simplicity and properties of this expression have even led to the extremely fruitful suggestion the MHV amplitudes be promoted to the underlying vertices of the theory [2].

At one-loop level the MHV n-gluon amplitudes have also been determined analytically. The different particle types contributing to the internal loop can be conveniently organized into supersymmetric multiplets. The $\mathcal{N} = 4$ amplitudes were constructed first [3] with the $\mathcal{N} = 1$ following shortly afterwards [4]. The remaining scalar has now finally succumbed to analytic attack [4–6].

Graviton scattering amplitudes are considerably more computationally complex. Expressions for the n-point MHV tree amplitudes were constructed from their soft and collinear factorizations [7] and subsequently proven to be correct in Ref. [8]. We will be following the approach of constructing rational terms from their soft and collinear factorizations in this Letter. To date, at one loop the only known all-n rational piece of a gravity amplitude is for the pure gravity ''all-plus'' case. The existence of a compact, all-n expression for the rational part of a less-than-maximally supersymmetric supergravity theory is significant, as it is a manifestation of a yet-to-be understood pattern of simplifications underpinning quantum gravity theories. These explicit expressions are a rich source of data for resolving this hidden structure.

For one-loop amplitudes there are more potential components which can be organized into $\mathcal{N} = 8$, 6, 4, 1, 0 matter contributions. The $\mathcal{N} = 8$ MHV contribution was first calculated in Ref. [9] where a remarkable similarity between these amplitudes and those of $\mathcal{N} = 4$ Yang-Mills theory was observed: the one-loop amplitudes are comprised entirely of scalar box integrals with rational coefficients. This ''no-triangle'' feature has been shown to extend to all one-loop $\mathcal{N} = 8$ amplitudes [10]. In terms of these matter contributions the $\mathcal{N} = 4$ supergravity one-loop amplitude is

$$M_n^{N=4} = M_n^{N=8} - 4M_n^{N=6,\text{matter}} + 2M_n^{N=4,\text{matter}}. \qquad (1)$$

Extensions to the basic $\mathcal{N} = 4$ theory can be obtained from variants of this formula.

One-loop amplitudes in gauge and gravity theories have an expansion in terms of scalar n-point integral functions I_n which encompass the transcendental functions together with a rational remainder R_n. Amplitudes involving massless particles contain scalar functions with $n = 2, 3, 4$, to order $O(\epsilon)$ in the dimensional regularization parameter. The symmetries of the specific theory reduce the general case to

$$M_n^{N=8} = \sum a_i I_4^i$$

$$M_n^{N=6} = \sum a_i I_4^i + \sum b_j I_3^j = \sum a_i I_4^{\text{tc}:i} + \sum b_k I_3^{3m:k}$$

$$M_n^{N=4} = \sum a_i I_4^{\text{tc}:i} + \sum b_j I_3^{3m:j} + \sum c_k I_2^k + R_n \qquad (2)$$

where I_4^{tc} is the specific combination of a scalar box integral, with its descendant scalar triangles, which is IR finite.

For the MHV helicity configurations the three mass triangles I_3^{3m} are absent and the sum of boxes is restricted to the ''two-mass-easy'' boxes and the one-mass boxes. For $\mathcal{N} = 8$ all boxes contribute whereas for $\mathcal{N} = 6$, 4 the boxes are restricted to the configurations where the negative helicities appear on opposite massive legs. The $a_i^{N=8}$ were first computed in Ref. [9].

$$a_i^{N=8} = \frac{(-1)^n}{8} \langle m_1 m_2 \rangle^8 h(a, M, b) h(b, N, a) \text{tr}(aMbN)^2 \qquad (3)$$

[Note the normalizations of the physical amplitude are $\mathcal{M}^{\text{tree}} = (\kappa/2)^{n-2} M^{\text{tree}}$, $\mathcal{M}^{1-\text{loop}} = (\kappa/2)^n M^{1-\text{loop}}$.] The ''half-soft'' functions $h(a, M, b)$ have the explicit form given in Ref. [9]. We are using the usual spinor products (see, e.g., [11]). The $a_i^{N=6,4}$ take the related form

$$a_i^{N=6,4} = \frac{(-1)^n}{8} \frac{\langle m_1 m_2\rangle^{8-2A}}{\langle ab\rangle^{2A}} h(a,M,b)h(b,N,a) \times \mathrm{tr}(aMbN)^2(\langle am_1\rangle\langle am_2\rangle\langle bm_1\rangle\langle bm_2\rangle)^A \quad (4)$$

with $A = 1$ for $\mathcal{N} = 6$ and $A = 2$ for $\mathcal{N} = 4$. The sum over bubbles includes bubbles where the clusters contain exactly one negative helicity leg and at least one positive helicity leg. The explicit form of the $c_i^{N=4}$ is given in the appendix of Ref. [11]. The a_i and c_i are, in general, computed using unitarity based techniques.

The rational terms R_n cannot be determined by four-dimensional unitarity. They may be determined using "D-dimensional" unitarity [12] but are thus more difficult to obtain. Explicit computations of R_n have been restricted to $n = 4, 5$ [13–15]. In the next section we present an all-n form of R_n thus completing the $\mathcal{N} = 4$ one-loop MHV amplitude.

Result.—The n-point rational term, which completes the n-point amplitude $M(m_1^-, m_2^-, p_1^+, \ldots, p_{n-2}^+)$ is

$$R_n = (-1)^n \frac{\langle m_1 m_2\rangle^4}{2}\left(R_n^0 + \sum_{r=3}^{n-2} R_n^r\right). \quad (5)$$

In the above

$$R_n^0 = \sum_{\text{boxes}} \frac{[ab]^2}{\langle ab\rangle^2} h(a,M,b)h(b,N,a) \times (\langle m_1 a\rangle\langle m_2 a\rangle\langle m_1 b\rangle\langle m_2 b\rangle)^2 \quad (6)$$

where there is a contribution for each box integral function present in the amplitude. The R_n^0 contain spurious quadratic singularities which are necessary to cancel those in the box integral contributions [11]. The remaining R_n^r are

$$R_n^r = \sum_{\text{subsets}} C_r[\{p\}^{(r)}]\hat{S}^{n-2-r}_{\{q\}^{(n-2-r)}}. \quad (7)$$

The sum is over subsets $\{p\}^{(r)}$ of $\{p_1, \ldots, p_{n-2}\}$ of length r of which there are $(n-2)!/r!/(n-2-r)!$. The $\{q\}^{(n-2-r)}$ are the remaining positive helicity legs. The C_r are

$$C_r[\{p_1, \cdots p_r\}] = \sum_{\text{perms}} \frac{[p_1p_2][p_2p_3]\cdots[p_rp_1]}{\langle p_1p_2\rangle\langle p_2p_3\rangle\cdots\langle p_rp_1\rangle} \quad (8)$$

where the sum over permutations is over the $(r-1)!$ cyclically independent choices of orderings of $\{p_1, \ldots, p_r\}$.

The $\hat{S}^m$ are polynomial in the objects $A[a;s]$,

$$A[a;s] \equiv \frac{[sa]\langle am_1\rangle\langle am_2\rangle}{\langle sa\rangle\langle sm_1\rangle\langle sm_2\rangle} \quad (9)$$

and are best defined by their soft behavior as $s \in \{q\}^m$ becomes soft,

$$\hat{S}^m_{\{q\}^m} \longrightarrow -\mathrm{soft}(s^+) \times \hat{S}^{m-1}_{\{q\}^m - s} \quad (10)$$

where $\mathrm{soft}(s^+)$ is the soft-factorization function [7],

$$\mathrm{soft}(n^+) = -\frac{1}{\langle 1n\rangle\langle nn-1\rangle}\sum_{j=2}^{n-2}\frac{\langle 1j\rangle\langle jn-1\rangle[jn]}{\langle jn\rangle} \quad (11)$$

together with the restriction that *any cyclic combinations of* $A[q_i, q_j]$ *are excluded*. Note that the negative sign is necessary since there is an overall factor of $(-1)^n$ in the amplitude. The $\hat{S}_m$ take the form

$$\hat{S}^m_{\{q\}^m} = \prod_{k=1}^m \hat{S}^1_{q_k} - \text{ cycle terms.} \quad (12)$$

The first few are given by, (note $\hat{S}^0 = 1$)

$$\begin{aligned}\hat{S}^1_{q_1} &= \sum_{p_j\in\{p\}^{(n-3)}} A[p_j;q_1]\\ \hat{S}^2_{\{q_1,q_2\}} &= \sum_{p_j\in\{p\}^{(n-4)}} A[p_j;q_1]\sum_{p_l\in\{p\}^{(n-4)}} A[p_l;q_2]\\ &\quad + A[q_1;q_2]\sum_{p_j\in\{p\}^{(n-4)}} A[p_j;q_1]\\ &\quad + A[q_2;q_1]\sum_{p_j\in\{p\}^{(n-4)}} A[p_j;q_2]\\ &= \hat{S}^1_{q_1}\hat{S}^1_{q_2} - A[q_1;q_2]A[q_2;q_1].\end{aligned} \quad (13)$$

The cyclic combinations of $A[q_i;q_j]$ simplify into cyclic combinations of $[q_iq_j]/\langle q_iq_j\rangle$, e.g.,

$$A[q_1;q_2]A[q_2;q_1] = \frac{[q_1q_2]^2}{\langle q_1q_2\rangle^2} \quad (14)$$

and are nonsingular in the soft limit. We will present an alternative description of R_n in the next section.

The structure of R_n is a rational function of the spinor variables λ_a^i and $\bar{\lambda}_{\dot{a}}^i$. The function is rational in λ_a^i but only polynomial in $\bar{\lambda}_{\dot{a}}^i$, the polynomial being homogeneous of degree $2(n-2)$. The tree MHV amplitude shares this feature but the polynomial is of degree $2(n-3)$. Consequently the R_n have an analogous "twistor-space" structure to the MHV tree amplitudes [16,17].

Construction.—The form of R_n was obtained from soft and collinear factorizations. Note that an MHV amplitude in a supergravity theory does not have any physical multi-particle poles. The collinear limit occurs when legs k_a and k_b are collinear, $k_a k_b \to 0$. Unlike Yang-Mills amplitudes, gravity amplitudes are not singular in the collinear limit, but acquire a "phase singularity" [9] that is specified in terms of amplitudes with one less external leg. If $k_a \to zK$ and $k_b \to (1-z)K$,

$$M_n(\cdots, a^{h_a}, b^{h_b}) \overset{a\|b}{\longrightarrow} \sum_{h'} \mathrm{Sp}^{h_a h_b}_{-h'} M_{n-1}(\cdots, K^{h'}) + F_n \quad (15)$$

where the h's denote the various helicities of the gravitons and F_n indicates the remainder term with no phase singularity. The nonzero "splitting functions" are [9]

$$\mathrm{Sp}_{-}^{++} = -\frac{[ab]}{z(1-z)\langle ab\rangle}, \qquad \mathrm{Sp}_{+}^{-+} = -\frac{z^3[ab]}{(1-z)\langle ab\rangle}. \tag{16}$$

Gravity amplitudes also have soft-limit singularities [7] as $k_n \to 0$,

$$M_n(\cdots, n-1, n^h) \overset{k_n\to 0}{\longrightarrow} \mathrm{soft}(n^h) M_{n-1}(\cdots, n-1). \tag{17}$$

An important result of Ref. [9] is that the splitting and soft-factorization functions (11) do not obtain loop corrections. The entire amplitude must satisfy these soft and collinear factorizations. With the exception of the collinear limit of two positive helicity legs, the transcendental functions and rational term factorize independently [11,14].

When considering these limits it is useful to use an alternate form:

$$\sum_{r=3}^{n-2} R_n^r = \sum_{r=3}^{n-2} \sum_{\mathrm{subsets}} \hat{C}_r[\{p_j\}] S^{n-r-2}_{\{q\}^{(n-r-2)}} \tag{18}$$

where

$$S^m_{\{q\}^m} = \prod_{k=1}^{m} \hat{S}^1_{q_k} \tag{19}$$

and

$$\hat{C}_r[\{p\}] = C_r[\{p\}] - \sum_{s=2}^{r-2} \sum_{\mathrm{subsets}} \epsilon_{r,s} C_{r-s}[\{b\}] \times C_s[\{c\}] + \cdots \tag{20}$$

where $\{c\}$ is a subset of $\{p\}$ of length s and $\{b\} = \{p\} - \{c\}$. The $\hat{C}_r$ is simply the weighted sum of all single and multiple cycles. $\hat{C}_3 = C_3$ and $\hat{C}_4 = C_4$ but

$$\hat{C}_5[\{a_1 \cdots a_5\}] = C_5[\{a_1 \cdots a_5\}] - \sum_{\mathrm{subsets}} C_2[\{a_1, a_2\}]] C_3[\{a_3, a_4, a_5\}] \tag{21}$$

or using simplified notation,

$$\begin{aligned}
\hat{C}_5 &= C_5 - C_2C_3\\
\hat{C}_6 &= C_6 - C_2C_4 - C_3C_3\\
\hat{C}_7 &= C_7 - C_2C_5 - 2C_3C_4 + C_2C_2C_3/2\\
\hat{C}_8 &= C_8 - C_2C_6 - 2C_3C_5 - C_4C_4 + C_2C_3C_3\\
&\quad + C_2C_2C_4/2.
\end{aligned} \tag{22}$$

In this form, the cycle terms previously subtracted from $\hat{S}^m$ lie with the $\hat{C}_r$ terms, leaving S^m which have simpler soft and collinear behavior. This form is useful in examining the soft and collinear limit but is really a more complicated expression where material has been added to both $\hat{S}_r$ and C_r.

The soft behavior of R_n^0 can be derived from the soft behavior of the half-soft functions [9],

$$h(a, M, b) \overset{k_m\to 0}{\longrightarrow} -\mathrm{soft}_m(a, M, b) h(a, M-m, b) \tag{23}$$

for legs $m \in M$. Where

$$\mathrm{soft}_m(a, M, b) = -\frac{1}{\langle am\rangle\langle mb\rangle} \sum_{j\in M-m} \frac{\langle aj\rangle\langle jb\rangle[jm]}{\langle jm\rangle}. \tag{24}$$

From this property of the half-soft functions we can show

$$R_n^0 \overset{k_n\to 0}{\longrightarrow} -\mathrm{soft}(n^+) \times R^0_{n-1}. \tag{25}$$

The soft behavior of S_1 is quite clear

$$S_q^1 \overset{k_q\to 0}{\longrightarrow} -\mathrm{soft}(q^+) \times 1, \qquad S_q^1 \overset{k_{\mathrm{other}}\to 0}{\longrightarrow} \text{finite} \tag{26}$$

so

$$S^m_{\{q\}^m} \overset{k_s\to 0}{\longrightarrow} -\mathrm{soft}(s^+) \times S^{m-1}_{\{q\}^m - s} \qquad s \in \{q\}^m. \tag{27}$$

The $\hat{C}_r$ do not contribute to any soft singularity so we can deduce

$$R_n^i \overset{k_p\to 0}{\longrightarrow} -\mathrm{soft}(p^+) \times R^i_{n-1}, \qquad i = 3, \ldots, n-3. \tag{28}$$

The term R_n^{n-2} has no soft singularity.

Collinear Limits.—There are three types of collinear limit. The amplitude vanishes as two negative legs become collinear as we would expect since the *daughter* amplitude $M(K^-, 3^+, \cdots, n^+)$ vanishes in any supersymmetric theory. There are two nonvanishing independent collinear limits—where the legs are (m^-, p^+) and (p_a^+, p_b^+). (Multicollinear limits for this case give no further constraints beyond iteratively applying two-particle collinear limits.)

First consider the limit (m^-, p^+). Note that R_n^{n-2} does not contribute to this limit. The function S_q^1 has no collinear phase singularity unless $q = p$, in which case

$$S_p^1 \overset{m\|p}{\longrightarrow} \frac{z^2[pm]}{z(1-z)\langle pm\rangle} = -\frac{1}{z^2}\mathrm{Sp}_{+}^{-+}. \tag{29}$$

So

$$S^t_{\{q\}^t} \overset{m\|p}{\longrightarrow} -\frac{1}{z^2}\mathrm{Sp}_{+}^{-+} \times S^{t-1}_{\{q\}^t - p} \tag{30}$$

for $p \in \{q\}^t$ and zero otherwise. The $\hat{C}_r$ do not contribute to this collinear limit and so we deduce

$$\langle mm'\rangle^4 R_n^i \overset{m\|p}{\longrightarrow} -\mathrm{Sp}_{+}^{-+}\langle Km'\rangle^4 R^i_{n-1} \tag{31}$$

with the factor of z^2 from $\langle mm'\rangle^4$ cancelling the z^{-2} in (29). The collinear limit of R_n^0 follows from the collinear behavior [9] of the half-soft functions

$$h(a,\{c,d,\cdots\},b)\overset{c\|d}{\longrightarrow}\frac{1}{z(1-z)}\frac{[cd]}{\langle cd\rangle}h(a,\{K,\cdots\},b) \quad (32)$$

from which we can deduce,

$$\langle mm'\rangle^4 R_n^0 \overset{m\|p}{\longrightarrow} -\mathrm{Sp}_{+}^{-+}\times\langle Km'\rangle^4 R_{n-1}^0. \quad (33)$$

The (p_a^+, p_b^+) collinear limit is a little more subtle. The terms in R_n^0 with a double phase singularity $\sim[ab]^2/\langle ab\rangle^2$ cancel exactly against the corresponding box integral contributions as $s_{ab}\to 0$ and give no phase singularity. The remaining terms in R_n^0 we refer to as $R_n^0|_{\mathrm{red}}$ and should satisfy:

$$R_n^0|_{\mathrm{red}}+\sum_{i=3}^{n-2}R_n^i \to \mathrm{Sp}_{-}^{++}\times\left(R_{n-1}^0+\sum_{i=3}^{n-3}R_{n-1}^i\right). \quad (34)$$

Note that this is the only factorization the term R_n^{n-2} contributes to. Although, at present, we have no analytic proof that the n-point expression has the correct collinear limit we have checked this numerically up to ten points. Note that, unlike the (m^-, p^+) collinear limit it is not satisfied "term-by-term" for the R_n^i but only by the total.

The expression for R_n gives a candidate amplitude which satisfies all physical collinear and soft factorizations, contains no spurious singularities, and satisfies the expected symmetries of the amplitude. We do not possess a proof that this expression is correct beyond five points although experience suggests it is extremely likely to be so: for the MHV tree amplitudes [7], the $\mathcal{N}=8$ MHV, and the all-plus one-loop amplitudes [9], soft and collinear constraints were sufficient to generate expressions which were subsequently proven correct. Indeed, there are recent suggestions [18] that soft limits alone may determine tree amplitudes.

Conclusions.—At present the perturbative structure of (super)gravity theories appears to be considerably more constrained with hidden structures and more symmetries than were apparent only a few years ago. The existence of explicit amplitudes is of key importance in forming and testing conjectures in perturbative field theory. Currently, very few explicit loop amplitudes exist to test perturbation theory beyond tree level in gravity theories. We have proposed an expression for the n-graviton MHV one-loop amplitude in $\mathcal{N}=4$ supergravity. This expression adds to a very small list of all-n one-loop expressions in gravity: the $\mathcal{N}=8$ and $\mathcal{N}=6$ MHV amplitudes and the pure gravity "all-plus" amplitude. Such explicit expressions have been extremely useful in the past in elucidating the perturbative structure of gauge theories. Our expression provides a goal for other approaches such as, for example, the gauge-gravity conjectures [15,19,20]. In general, we hope that this series of amplitudes will prove useful in untangling the perturbative expansion of quantum (super) gravity.

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