Universal Nonlinear Small-Scale Dynamo

A. Beresnyak
Los Alamos National Laboratory, Los Alamos, New Mexico, 87545, USA
Ruhr-Universität Bochum, 44780 Bochum, Germany
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We consider astrophysically relevant nonlinear MHD dynamo at large Reynolds numbers (Re). We argue that it is universal in a sense that magnetic energy grows at a rate which is a constant fraction C_E of the total turbulent dissipation rate. On the basis of locality bounds we claim that this ''efficiency of the small-scale dynamo'', C_E, is a true constant for large Re and is determined only by strongly nonlinear dynamics at the equipartition scale. We measured C_E in numerical simulations and observed a value around 0.05 in the highest resolution simulations. We address the issue of C_E being small, unlike the Kolmogorov constant which is of order unity.

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Introduction.—MHD turbulence is ubiquitous in astrophysical and space environments [1]. Reynolds numbers are, typically, very high, owing to astrophysical scales which are enormous compared to dissipative scales. One of the central processes of MHD dynamics is how conductive fluid generates its own magnetic field, a process known broadly as ''dynamo.'' Turbulent dynamo has been subdivided into large-scale (mean-field) dynamo and small-scale (fluctuation) dynamo depending on whether magnetic fields are amplified on scales larger or smaller than outer-scale of turbulence. Although several ''no-dynamo'' theorems have been proved for flows with symmetries, a generic turbulent flow, which possesses no exact symmetry, was expected to amplify magnetic field by stretching, due to the particle separation in a turbulent flow. For the large-scale dynamo, a ''twist-stretch-fold'' mechanism was introduced [2]. Turbulent flow possessing perfect statistical isotropy can not generate large-scale field, so the observed large-scale fields, such as in the disk galaxies, are generated when statistical symmetries of turbulence are broken by large-scale asymmetries of the system, such as stratification, rotation and shear (see, e.g., [3]). Since these symmetries are only weakly broken, large-scale dynamo is slow. Small-scale dynamo does not suffer from this restriction and can be fast. Kinematic small-scale dynamo, which ignores the backreaction of the magnetic field has been studied extensively [4]. However, from these models it was not clear whether after kinematic stage it will continue to operate. Also for astrophysically large Re it becomes inapplicable at very short time scales. Indeed, kinematic dynamo possessing positive spectral index, typically $3/2$, is incompatible with observations in galaxy clusters [5] which clearly indicate steep spectrum with negative power index at small scales. Because of preexisting astrophysical fields, small-scale dynamo starts in nonlinear regime. It was discovered numerically that small-scale dynamo continues to grow after kinematic stage, producing steep spectrum at small scales and significant outer-scale fields [6–9]. Furthermore, MHD turbulence produces turbulent diffusivity (aka ''β effect''), which is essential for large-scale dynamo [3] and reconnection [10,11]. Saturation of small-scale dynamo seems to be independent on Re and Pr as long as Re is large [6] and the magnetic energy growth rate could be constant [8,9,12,13]. Small-scale dynamo is faster than large-scale dynamo in most astrophysical environments and magnetic energy grows quickly to equipartition with kinetic motions, with the largest scales of such field being a fraction of the outer scale of turbulence. Subsequently, these turbulent fields are slowly ordered by mean-field dynamo, with turbulent diffusivity of MHD turbulence playing essential role. In this Letter we provide sufficient analytical and numerical argumentation behind the universality of the nonlinear small-scale dynamo.

Nonlinear small-scale dynamo.—We assume that the spectra of magnetic and kinetic energies at a particular moment of time are similar to what is presented on Fig. 1. Magnetic and kinetic spectra cross at some ''equipartition'' scale $1/k^*$, below which both spectra are steep due to MHD cascade (see, e.g., [14,15]). This is suggested by both numerical evidence [9,16] and observations of magnetic fields in clusters of galaxies [5]. At larger scales magnetic spectrum is shallow, k^α, $\alpha > 0$, while kinetic spectrum is steep due to the hydrodynamic cascade. Most of the magnetic energy is concentrated at scale $1/k^*$. We designate C_K and C_M as Kolmogorov constants of hydro and MHD, respectively. The hydrodynamic cascade rate is ϵ and the MHD cascade rate as ϵ_2. Because of the conservation of energy in the inertial range, magnetic energy will grow at a rate $\epsilon - \epsilon_2$. We will designate $C_E = (\epsilon - \epsilon_2)/\epsilon$ as an ''efficiency of the small-scale dynamo'' and will argue that this is a true constant, since: (a) turbulent dynamics is local in scale in the inertial range; (b) neither ideal MHD nor Euler equations contain any scale explicitly. Magnetic energy, therefore, grows linearly with time if $\epsilon = \text{const}$. The equipartition scale $1/k^*$ will grow with time as $t^{3/2}$

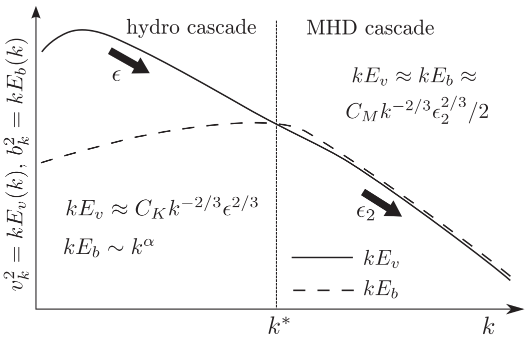

FIG. 1. A cartoon of kinetic and magnetic spectra in small-scale dynamo, at a particular moment of time when equipartition wave number is k^*.

[13]. This is equivalent to saying that small-scale dynamo saturates at several dynamical times at scale $1/k^*$ and proceeds to a twice larger scale [12]. If magnetic energy grows approximately till equipartition [6,9], the whole process will take around several dynamical time scales of the system, or more quantitatively, $(C_K^{3/2}/C_E)(L/v_L)$.

Locality of the small-scale dynamo.—We will use "smooth filtering" approach with dyadic-wide filter in k space [17]. We designate a filtered vector quantity as $\mathbf{a}^{[k]}$ where k is a center of a dyadic Fourier filter in the range of wave numbers $[k/2, 2k]$. The actual logarithmic width of this filter is irrelevant to further argumentation, as long as it is not very small. We will assume that the vector field $\mathbf{a}$ is Hölger-continuous with some exponent and designate $a_k = \langle|\mathbf{a}^{[k]}|^3\rangle^{1/3}$ which has to scale as k^{σ_3}, e.g., $k^{-1/3}$ for velocity in Kolmogorov turbulence. The energy cascade rate is $\epsilon = C_K^{-3/2} k v_k^3$, where we defined Kolmogorov constant C_K by third order, rather than second order quantities. We will keep this designation, assuming that traditional Kolmogorov constant could be used instead. We use spectral shell energy transfer functions such as $T_{vv}(p,k) = -\langle \mathbf{v}^{[k]}(\mathbf{v}\cdot\nabla)\mathbf{v}^{[p]}\rangle$, $T_{w^+w^+}(p,k) = -\langle \mathbf{w}^{+[k]}(\mathbf{w}^-\cdot\nabla)\mathbf{w}^{+[p]}\rangle$ [18], applicable to incompressible ideal MHD equations, where $w^{\pm}$ are Elsässer variables and v, b and $w^{\pm}$ are measured in the same Alfvenic units. Using central frequency k and studying "infrared" (IR) transfers from $p \ll k$, and "ultraviolet" (UV) transfers, from $q \gg k$, we will provide absolute bounds on $|T|$, in units of energy rate as in [17,19], and *relative* volume-averaged bounds which are divided by the actual energy rate and are dimensionless. We will consider three main k intervals presented on Fig. 1: $k \ll k^*$ ("hydrodynamic cascade"), $k \sim k^*$ (dynamo) and $k \gg k^*$ ("MHD cascade").

MHD cascade, $k \gg k^*$.—The only energy cascades here are Elsässer cascades and, by the design of our problem, w^+ and w^- have the same statistics, so we will drop $\pm$. For an exchange with $p \ll k$ band, for $|T_{ww}|$, using Hölger inequality and wave number conservation we get an upper bound of $pw_p w_k^2$ and for $q \gg k$ band it is $kw_q^2 w_k$, these bounds are asymptotically small. For the full list of

TABLE I. Transfers and upper limits.

Transfers	$p \ll k$	$q \gg k$
$T_{vv}(p,k) = -\langle \mathbf{v}^{[k]}(\mathbf{v}\cdot\nabla)\mathbf{v}^{[p]}\rangle$	$pv_p v_k^2$	$kv_k v_q^2$
$T_{bb}(p,k) = -\langle \mathbf{b}^{[k]}(\mathbf{v}\cdot\nabla)\mathbf{b}^{[p]}\rangle$	$pb_p v_k b_k$	$kb_k v_q b_q$
$T_{vb}(p,k) = \langle \mathbf{b}^{[k]}(\mathbf{b}\cdot\nabla)\mathbf{v}^{[p]}\rangle$	$pv_p b_k^2$	$kb_k v_q b_q$
$T_{bv}(p,k) = \langle \mathbf{v}^{[k]}(\mathbf{b}\cdot\nabla)\mathbf{b}^{[p]}\rangle$	$pb_p v_k b_k$	$kv_k b_q^2$
$T_{w^+w^+}(p,k) = -\langle \mathbf{w}^{+[k]}(\mathbf{w}^-\cdot\nabla)\mathbf{w}^{+[p]}\rangle$	$pw_p w_k^2$	$kw_k w_q^2$

transfers and limits refer to Table I. The relative bound should be taken with respect to $C_M^{-3/2} k w_k^3$, where C_M is a Kolmogorov constant for MHD, from which we get that most of the energy transfer with the $[k]$ band should come from $[kC_M^{-9/4}, kC_M^{9/4}]$ band, see [15]. The global transfers between kinetic and magnetic energy must average out in this regime, nevertheless, the pointwise IR and UV transfers can be bounded by $pb_p v_k b_k$ and $kb_q^2 v_k$ and are small [19].

Hydrodynamic cascade, $k \ll k^*$.—Despite having some magnetic energy at these scales, most of the energy transfer is dominated by velocity field. Indeed, $|T_{vv}|$ is bounded by $pv_p v_k^2$ for $p \ll k$ and by $kv_q^2 v_k$ for $q \gg k$. Compared to these, $|T_{bv}|$ transfers are negligible: $pb_p v_k b_k$ and $kb_q^2 v_k$. For magnetic energy in $p \ll k$ case we have $|T_{vb}|$ and $|T_{bb}|$ transfers bounded by $pv_p b_k^2$, $pb_p v_k b_k$ and for $q \gg k$ case $|T_{vb}|$ and $|T_{bb}|$ are bounded by $kb_k v_q b_q$. Out of these three expressions the first two go to zero, while the third goes to zero if $\alpha - 2/3 < 0$ or have a maximum at $q = k^*$ if $\alpha - 2/3 > 0$. This means that for the transfer to magnetic energy we have IR locality, but not necessarily UV locality. Note that magnetic energy for $k \ll k^*$ is small compared to the total, which is dominated by $k = k^*$. We will assume that $\alpha - 2/3 > 0$ and that the spectrum of b_k for $k < k^*$ is formed by nonlocal $|T_{vb}|$ and $|T_{bb}|$ transfers from k^*, namely, magnetic structures at k are formed by stretching of magnetic field at k^* by velocity field at k. Magnetic spectrum before k^* is, therefore, nonlocal and might not be a power-law, but our further argumentation will only require that $b_k < v_k$ for $k < k^*$.

Dynamo cascade $k = k^*$.—In this transitional regime our estimates of Elsässer UV transfer and kinetic IR transfer from two previous sections will hold. We are interested how these two are coupled together and produce observed magnetic energy growth. IR $p \ll k^*$ $|T_{vb}|$ and $|T_{bb}|$ transfers will be bounded by $pv_p b_{k^*}^2$ and $pb_p v_{k^*} b_{k^*}$, which go to zero, so there is a good IR locality. Ultraviolet transfers will be bounded by $k^* b_{k^*} b_q v_q$. This quantity also goes to zero as q increases, so there is an UV locality for this regime as well. Let us come up with bounds of relative locality. Indeed, the actual growth of magnetic energy was defined as $\epsilon_B = \epsilon - \epsilon_2 = C_E C_K^{-3/2} k v_k^3$. So, $p \ll k^*$ IR bound is $k^* C_E^{3/2} C_K^{-9/4}$ and UV bound is $k^* C_E^{-3/2} C_M^{9/4}$. We conclude that most of the interaction which result in magnetic energy growth must reside in the wave vector interval

TABLE II. Three-dimensional MHD simulations.

Run	n	N^3	Dissipation	$\langle\epsilon\rangle$	Re	C_E
M1-6	6	256^3	$-7.6\times10^{-4}k^2$	0.091	1000	0.031 ± 0.002
M7-9	3	512^3	$-3.0\times10^{-4}k^2$	0.091	2600	0.034 ± 0.004
M10-12	3	1024^3	$-1.2\times10^{-4}k^2$	0.091	6600	0.041 ± 0.005
M13	1	1024^3	$-1.6\times10^{-9}k^4$	0.182	$\cdots$	0.05 ± 0.005
M14	1	1536^3	$-1.5\times10^{-15}k^6$	0.24	$\cdots$	0.05 ± 0.005

of $k^*[C_E^{3/2}C_K^{-9/4}, C_E^{-3/2}C_M^{9/4}]$. Numerically, if we substitute $C_K = 1.6$, $C_M = 4.2$, $C_E = 0.05$ we get the interval of $k^*[0.004, 2000]$. So, despite being asymptotically local, small-scale dynamo can be fairly nonlocal in practice.

Summarizing, the kinetic cascade at large scales and the MHD cascade at small scales are dominated by local interactions. The transition between the kinetic cascade and the MHD cascade is also dominated by local interactions, and since ideal MHD equations do not contain any scale explicitly, the efficiency of small-scale dynamo C_E is a true universal constant. Note that C_E relates energy fluxes, not energies, so this claim is unaffected by the presence of intermittency. Magnetic spectrum at $k \ll k^*$ is dominated by nonlocal triads that reprocess magnetic energy from $k = k^*$ but, since this part of the spectrum contains negligible magnetic energy, our universality claim is unaffected by this nonlocality.

Numerical results.—We performed numerical simulations of statistically homogeneous isotropic small-scale dynamo by solving MHD equations with stochastic nonhelical driving and explicit dissipation with $Pr_m = 1$. The details of the code and driving are described in detail in our earlier publications [16,20] and Table II shows simulation parameters. We started each simulation from previously well-evolved driven hydro simulation by seeding low level white noise magnetic field. We ran several statistically independent simulations in each group and obtained growth rates and errors from sample averages. In all simulations, except M14, the energy injection rate was controlled. Figure 2 shows sample-averaged time evolution of magnetic energy. Growth is initially exponential and smoothly transition into the linear stage. Note, that scatter is initially small, but grows with time, which is consistent with the picture of magnetic field growing at progressively larger scales and having progressively less independent realizations in a single data cube.

Efficiency of small-scale dynamo.—Our C_E is much smaller than unity. One would expect a quantity of order unity because this is a universal number, determined only by strong interaction on equipartition scale. If we refer to the ideal incompressible MHD equations, written in terms of Elsässer variables, $\partial_t \mathbf{w}^\pm + \hat{S}(\mathbf{w}^\mp \cdot \nabla)\mathbf{w}^\pm = 0$, the dynamo could be understood as decorrelation of $\mathbf{w}^\pm$ which are originally equal to each other in the hydrodynamic cascade. In our case this decorrelation is happening at the equipartition scale k^*. Being time dependent, it propagates upscale, while ordinarily energy cascade goes downscale. The small value of C_E might be due to this. As opposed to picture with multiple reversals and dissipation due to microscopic diffusivity, typical for kinematic case, in our picture we appeal to *turbulent diffusion* which helps to create large-scale field. Both stretching and diffusion depend on turbulence at the same designated scale $1/k^*$, so in the asymptotic regime of large Re one of these processes must dominate. As C_E is small, stretching and diffusion are close to canceling each other.

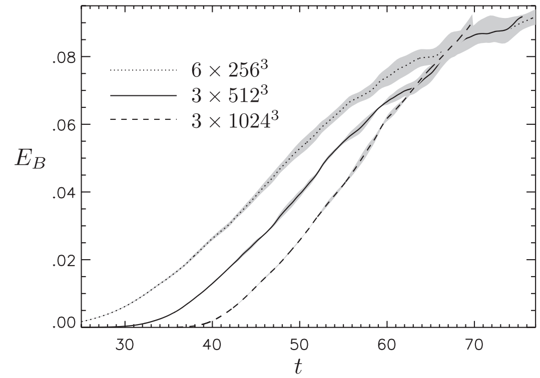

FIG. 2. Magnetic energy growth vs time in code units, observed in simulations run M1-6 ($\tau_\eta = 0.091$ in code units), run M7-9 ($\tau_\eta = 0.057$), and run M10-12 ($\tau_\eta = 0.036$). We used sample averages which greatly reduced fluctuations and allowed us to measure C_E with sufficient precision.

Kinematic dynamo rates.—A better studied and understood kinematic dynamo might shed some light on the problem of small C_E. In the kinematic regime, when we neglect Lorentz force in the MHD equation, the growth is exponential and the rate is expected to come from fastest shearing rate of smallest turbulent eddies. Observed rates, however, are smaller which was interpreted as competition between stretching and turbulent mixing [22]. In our simulations, in the kinematic regime of run M7-9 we observed growth rate $\gamma\tau_\eta = 0.0326$, where $\tau_\eta = (\nu/\epsilon)^{1/2}$ is a Kolmogorov time scale, which is consistent with [6,23]. In terms of minimum time scale, $\tau_{\min} \approx 9\tau_\eta$, $\gamma\tau_{\min} = 0.3$, which is still small. Kazantsev-Kraichnan model [4] predicts $\gamma\tau_{\min} \sim 1$. This model, however, uses ad-hoc delta-correlated velocity which does not correspond to any dynamic turbulence and its statistics is time-reversible as opposed to time-irreversible real turbulence. Time irreversibility of hydro turbulence actually mandates that fluid particles separate faster backwards in time, since $\langle v_{\|l}^3\rangle = -4/5\epsilon l$ is negative.

In order to study the interplay of stretching and diffusion, we performed several simulations of kinematic dynamo forward and backward in time. We followed full three-dimensional evolution of v and b and approximated "backward in time" by reversing velocity direction. Initial condition for magnetic field was typically random noise. Since we could not reverse viscous losses in DNS, we used

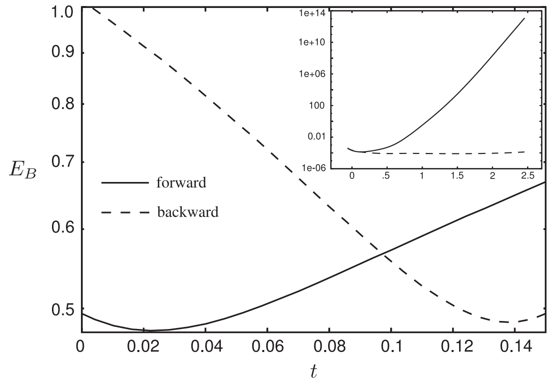

FIG. 3. Evolution of magnetic energy forward and backward in time (dashed is with reversed x axis). Inset: a naive simulation with initial state $+\mathbf{v}$ and $-\mathbf{v}$ from Navier-Stokes simulation (not a rigorous backward in time simulation).

viscosity $\nu = 0$, but magnetic diffusivity $\eta > 0$. In the first set of simulations we set initial velocity as v and $-v$ from evolved viscous runs. The growth rates are shown on inset of Fig. 3. Quite surprisingly, the "backward" simulation did not produce any growth for several dynamical times. Unexpectedly, simply reversing velocity has such a profound impact on kinematic dynamo, despite spectra being very close to each other, suggesting that it is not only the spectrum that determines growth but rather the actual statistical properties of velocity, which will determine whether stretching or diffusion wins, i.e., if there is a dynamo or a no-dynamo, even in the simple kinematic case. In this simulation we observed a typical k^2 "thermal pool" at the end of velocity spectrum which had shortest time scales. "Thermal pool" was clearly time-irreversible, unlike the true physical thermal pool, consistent with [24].

The next series of simulations were reproducing an actual backward in time dynamics. In order to achieve this we evolved initial state for a fairly short time with $\nu = 0$ and then we evolved it for the same time reverting velocity with $\nu = 0$ and confirmed that final state is close to initial state, due to reversibility of truncated Euler equations. The results for dynamo growth is shown of Fig. 3. We see that backward dynamo is faster by a factor of 2.0 ± 0.1, which is actually consistent with the ratio of particle diffusion forward and backward in time in [22]. This result again reinforces our statement that dynamo is a result of competing mechanisms of turbulent stretching and turbulent diffusion and the outcome depends on statistics of velocity other than just velocity spectrum.

A different picture was suggested in high Pr case by [21], where, unlike our picture, magnetic energy was at scales much smaller than kinetic and an unspecified portion of kinetic energy was dissipated in hydrodynamic cascade, while the rest diverted into magnetic spectrum, then contributing $\sim 16\%$ to the magnetic growth, so C_E was between 0 and 0.16. Linear growth of the ratio of magnetic to kinetic energy with the rate of $0.0088(t/t_{\mathrm{eddy}})$ was proposed in [8], which is hard to compare to our result, since no exact relation between kinetic energy and the dissipation rate is available.

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