Self-Consistent Evolution of Magnetic Fields and Chiral Asymmetry in the Early Universe

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We show that the evolution of magnetic fields in a primordial plasma, filled with standard model particles at temperatures $T \gtrsim 10$ MeV, is strongly affected by the chiral anomaly—an effect previously neglected. Although reactions, equilibrating left and right electrons, are in thermal equilibrium for $T \lesssim 80$ TeV, a left-right asymmetry develops in the presence of strong magnetic fields. This results in magnetic helicity transfer from shorter to longer scales and lepton asymmetry present in the plasma until $T \sim 10$ MeV, which may strongly affect many processes in the early Universe.

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Magnetic fields are expected to play an important role in the early Universe. Recent observational indications of the presence of magnetic fields in the intergalactic medium [1–3] suggest that cosmological magnetic fields (CMFs) may survive even until the present epoch. Thus, they could have played the role of seeds for the formation of galactic magnetic fields. A number of mechanisms for the creation of CMFs at very high temperatures have been proposed (see, e.g., [4–6], and references therein).

In this Letter, we concentrate, however, on a different problem: We assume that strong CMFs were already generated at a temperature $\gtrsim 100$ GeV, and we study the subsequent evolution of such fields. Usually, this evolution is described by the system of Maxwell plus Navier-Stokes equations (for a detailed review, see [5,7]). Here we will argue that, for temperatures $T \gtrsim 10$ MeV, this system of MHD equations should be extended to include a new effective degree of freedom, even if all particles and reactions are described by just the standard model of particle physics. This significantly affects the evolution of CMFs and the state of the primordial plasma.

At such temperatures, rates of all perturbative processes related to the electron's finite mass are suppressed as $(m_e/T)^2$. Ignoring these corrections for a moment, the number of left- and right-chiral electrons [8] is conserved independently [9]. That is, apart from the vector current $j^\mu = \bar{\psi}\gamma^\mu\psi$ describing conservation of electric charge $(n_L + n_R)$, the average number density of the left- (right-) chiral electrons $n_{L,R} = \frac{1}{2V}\int d^3x\psi^\dagger(1 \pm \gamma_5)\psi$ does not change with time. This is true on time scales smaller than the chirality-flipping scale Γ_f^{-1}. Although the chirality-flipping rate is suppressed as compared to the rate of chirality-preserving weak and electromagnetic processes, it is faster than the Hubble expansion rate $H(T)$ for temperatures below 80 TeV [10], and chirality-flipping processes are in thermodynamic equilibrium. Yet, on time scales $\Gamma^{-1}_{\mathrm{EM,weak}} < t < \Gamma_f^{-1}$, one should introduce independent chemical potentials μ_L and μ_R for two approximately conserved number densities, with $n_{L,R} = \frac{\mu_{L,R}}{6}T^2$. In the presence of external classical fields, the conservation of the axial current is spoiled, however, by the chiral anomaly [11]—a quantum effect leading to a change of $n_L - n_R$:

$$\frac{d(n_L - n_R)}{dt} = \frac{2\alpha}{\pi}\frac{1}{V}\int d^3xE\cdot B = -\frac{\alpha}{\pi}\frac{d\mathcal{H}}{dt}, \qquad (1)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant and $\mathcal{H}$ is the magnetic helicity defined as

$$\mathcal{H}(t) = \frac{1}{V}\int_V d^3xA\cdot B \qquad (2)$$

(where B is the magnetic field and A the vector potential, with $B = \nabla\times A$). The quantity (2) is gauge invariant, provided that B is parallel to the boundary of V (see, e.g., [7]). The time evolution of $\mathcal{H}(t)$ is given by [7]

$$\frac{d\mathcal{H}}{dt} = -\frac{2}{V}\int_V d^3xE\cdot B. \qquad (3)$$

In terms of the difference of left and right chemical potentials, $\Delta\mu \equiv \mu_L - \mu_R$, Eq. (1) reads

$$\frac{d(\Delta\mu)}{dt} = -\frac{c_\Delta\alpha}{T^2}\frac{d\mathcal{H}(t)}{dt}, \qquad (4)$$

where c_Δ is a numerical coefficient of order one that describes the dependence of n_L on globally conserved charges in the primordial plasma.

If $\Delta\mu \neq 0$, the chiral anomaly leads to an additional contribution to the current in Maxwell's equations [12–19]:

$$\nabla\times B = \sigma E + \frac{\alpha}{\pi}\Delta\mu(t)B,$$

or, by combining it with the Bianchi identity $\nabla\times E = -\dot{B}$:

$$\frac{\partial B}{\partial t} = \frac{1}{\sigma}\nabla^2 B + \frac{\alpha}{\pi}\frac{\Delta\mu}{\sigma}\nabla\times B. \tag{5}$$

As weak reactions are fast enough at these temperatures to establish local thermodynamic equilibrium, in the background of long-wavelength electromagnetic fields, space-dependent chemical potentials $\mu_{L,R}(x)$ may be defined. Equations (1) and (4) can then be written in a local form, and Eq. (5) acquires additional terms, proportional to the gradients of $\Delta\mu(x)$ [15,17,18]. We assume fields to be slowly varying and neglect these effects as well as those depending on the velocity field. We will show that, even in this limit, the evolution of magnetic fields significantly changes as compared to the usual Maxwell equations. A more realistic analysis should include all the derivative terms, as well as the Navier-Stokes equation describing, in particular, turbulent effects known to be important for the evolution of CMFs. We leave a more complete microscopic derivation and an analysis of the full system to future work and use the simple model described above to illustrate the previously neglected effects.

Equations (4) and (5) remain valid in an expanding universe if written in conformal coordinates (see, e.g., [5,13,20]). Henceforth, we use conformal quantities and define conformal time as $\eta = \frac{M_*}{T}$, where $M_* = \sqrt{90/8\pi^3 g_*}M_{\rm Pl}$ and g_* is the effective number of relativistic degrees of freedom.

Equations (4) and (5) are translation and rotation invariant. We introduce the magnetic helicity density $\mathcal{H}_k$ and the magnetic energy density ρ_k in Fourier space, with $\rho_B(\eta) = \int dk\rho_k(\eta)$ and $\mathcal{H}(\eta) = \int dk\mathcal{H}_k(\eta)$ [21]. The quantities $\mathcal{H}_k$ and ρ_k obey the inequality $|\mathcal{H}_k| \le \frac{2}{k}\rho_k$, which is saturated for field configurations known as maximally helical fields. In our subsequent analysis, we focus on this case and choose for definiteness $\mathcal{H}_k > 0$ and $\Delta\mu > 0$. Multiplying the Fourier version of Eq. (5) by the complex-conjugate mode $\vec{B}_k^*$, we obtain, after some simple manipulations (see [24], Sec. E for details; cf. [22]),

$$\frac{\partial\mathcal{H}_k}{\partial\eta} = -\frac{2k^2}{\sigma_c}\mathcal{H}_k + \frac{\alpha}{\pi}\frac{k\Delta\mu}{\sigma_c}\mathcal{H}_k, \tag{6}$$

$$\frac{d(\Delta\mu)}{d\eta} = -(c_\Delta\alpha)\int dk\frac{\partial\mathcal{H}_k}{\partial\eta} - \Gamma_f\Delta\mu, \tag{7}$$

where we have restored the chirality-flipping rate Γ_f in Eq. (7) and used the conductivity $\sigma_c \equiv \sigma(\eta)/T \approx 70$ [25].

The system (6) and (7) has been previously studied in two regimes. It was demonstrated in Refs. [13,18,26] that, in the presence of a large initial chemical potential difference $\Delta\mu(\eta) > 0$, the quantity

$$\mathcal{H}_k(\eta) = \mathcal{H}_k^0 \exp\left\{\frac{2k}{\sigma_c}\left(\frac{\alpha}{2\pi}\int_{\eta_0}^{\eta}\Delta\mu(\tilde\eta)d\tilde\eta - k(\eta-\eta_0)\right)\right\} \tag{8}$$

grows exponentially fast for sufficiently long wavelengths. Conversely, in Ref. [15], the initial background of helical (hyper)magnetic fields was used to generate a nonzero chemical potential for $T > 100$ GeV.

In this work, however, we consider helical CMFs with some initial spectrum $\mathcal{H}_k^0$, already present at $T \sim 100$ GeV in the hot plasma, filled with particles in thermal equilibrium (cf. [20,22,23,27]). It was believed that, as $\Gamma_f \gg H(T)$ for $T \lesssim 80$ TeV, no chiral asymmetry will survive.

Chirality evolution.—Below, we show that both $\Delta\mu$ and the magnetic helicity do survive below 100 GeV on time scales much longer than diffusion or chirality-flipping times (until 10–100 MeV). [For $\Gamma_f \to 0$, the system (6) and (7) can even reach a stationary state with nonzero B and $\Delta\mu$.]

To see this, it is convenient to separate on the right-hand side of Eq. (7) a source term $S_B(\eta)$ (independent of $\Delta\mu$) (cf. [15]):

$$\frac{d(\Delta\mu)}{d\eta} = -[\Gamma_B(\eta) + \Gamma_f]\Delta\mu + S_B(\eta), \tag{9}$$

where

$$\Gamma_B(\eta) \equiv \frac{c_\Delta\alpha^2}{\pi\sigma_c}\int dkk\mathcal{H}_k = \frac{2c_\Delta\alpha^2}{\pi\sigma_c}\rho_B, \qquad S_B(\eta) \equiv 2\frac{c_\Delta\alpha}{\sigma_c}\int dkk^2\mathcal{H}_k. \tag{10}$$

We begin our analysis of Eqs. (6), (9), and (10) with the case where $\Gamma_f = 0$ and the field is initially "monochromatic," i.e.,

$$\mathcal{H}_k(\eta) = \mathcal{H}(\eta)\delta(k-k_0). \tag{11}$$

The form (11) is preserved during the evolution as Eq. (6) is homogeneous [28]. Putting in Eq. (9) $d(\Delta\mu)/d\eta = 0$ and $\Gamma_f = 0$, we find the so-called tracking solution:

$$\Delta\mu_{\rm tr} = \frac{S_B(\eta)}{\Gamma_B(\eta)} = \frac{2\pi k_0}{\alpha}. \tag{12}$$

This is an exact static solution of the system (6) and (9): $\Delta\mu_{\rm tr}$ and $\mathcal{H}(\eta)$ remain constant; i.e., dissipation due to magnetic diffusion is exactly compensated by growth due to a nonvanishing chemical potential difference $\Delta\mu_{\rm tr}$ [cf. (8)].

Until now, we have completely neglected the massiveness of the electrons. It is straightforward to compute that the rate $\Gamma_f(\eta)$ due to electromagnetic processes is [29] $\Gamma_f(\eta) \approx \alpha^2(\frac{m_e}{3M_*})^2\eta^2$. Equations (6) and (9) can be rewritten to describe deviations from the equilibrium static solution (12):

$$\frac{d\Delta\mu}{d\eta} = -\Gamma_B(\Delta\mu - \Delta\mu_{\rm tr}) - \Gamma_f\Delta\mu, \tag{13}$$

$$\frac{d\Gamma_B}{d\eta} = \frac{\Gamma_B}{\eta_\sigma}\left(\frac{\Delta\mu}{\Delta\mu_{\rm tr}} - 1\right), \tag{14}$$

where $\eta_\sigma \equiv \frac{2k^2}{\sigma_c}$ is the magnetic diffusion time. From Eq. (13), we see that Γ_B and Γ_f, which enter symmetrically in Eq. (9), play very different roles. The rate Γ_f, that depends only on temperature, constantly drives $\Delta\mu$ to zero. The Γ_B term pushes the system towards the equilibrium value (12) (that depends only on k_0). It depends on ρ_B and has its own dynamics [Eq. (14)].

If the magnetic field is large (such that $\Gamma_B \gg \Gamma_f$), any initial value of $\Delta\mu$ will be quickly "forgotten" and $\Delta\mu$ will be driven towards $\Delta\mu_{\rm tr}$. At that moment, a new tracking solution will take over, with $\Delta\mu - \Delta\mu_{\rm tr} \approx \gamma\Delta\mu_{\rm tr}$, where

$$\gamma(\eta) \equiv \frac{\Gamma_f(\eta)}{\Gamma_B(\eta)}. \tag{15}$$

This new solution is valid, provided two conditions hold: (i) $\gamma \ll \Gamma_B t$ and (ii) $\gamma \ll \Gamma_B \eta_\sigma$. When this holds, the evolution of Γ_B is given by (14)

$$\frac{d\Gamma_B}{d\eta} = -\frac{\gamma(\eta)}{\eta_\sigma}\Gamma_B = -\frac{1}{\eta_\sigma}\Gamma_f(\eta). \tag{16}$$

Equation (16) shows that Γ_B remains practically constant when $\eta \ll \eta_\sigma/\gamma(\eta)$, which is significantly longer than η_σ, as $\gamma \ll 1$. To estimate the time at which the function $\gamma(\eta) \sim 1$, we note that it evolves with time because of an increasing chirality-flipping rate $\Gamma_f(\eta) \propto \eta^2$ and because the total magnetic energy dissipates (16). Neglecting this latter change, we estimate γ to be given by

$$\gamma = \frac{\pi\sigma_c}{2c_\Delta}\left(\frac{m_e}{3M_*}\right)^2 \frac{\eta^2}{\frac{\pi^2}{30}g_* r_B} = \frac{10^{-5}}{r_B}\left(\frac{100\,\text{MeV}}{T}\right)^2\left(\frac{30}{g_*}\right), \tag{17}$$

where we used $r_B \equiv \rho_B/(\frac{\pi^2}{30}g_* T^4)$ as the fraction of magnetic energy density to the total energy density. From Eq. (16), we see in addition that Γ_B remains approximately constant as long as $\frac{1}{\eta_\sigma}\int\gamma(\eta)d\eta < 1$. Using (17), we find that (this is illustrated in Fig. A1 in Ref. [24])

$$\frac{\gamma(\eta)\eta}{3\eta_\sigma} \le 1. \tag{18}$$

Inverse cascade.—So far we have considered a toy model example of a monochromatic helical field (11). Although Eq. (6) is linear, the modes $\mathcal{H}_k$ are not independent for different k [due to the integral in Eq. (7)]. For a continuous spectrum, this interaction results in another very important effect: The initial spectrum reddens with time, the total helicity being conserved (similarly to the "inverse cascade" phenomenon [7], Sec. 7.2.3).

Indeed, let us consider first the case of two modes $(k_1, \mathcal{H}_1(\eta))$ and $(k_2, \mathcal{H}_2(\eta))$ with $k_1 > k_2$, to understand the situation qualitatively. While $\Gamma_B \gg \Gamma_f$, the evolution for $\Delta\mu$ has the form

$$\frac{d(\Delta\mu)}{d\eta} = -\frac{c_\Delta\alpha^2}{\pi\sigma_c}(k_1\mathcal{H}_1 + k_2\mathcal{H}_2)\Delta\mu + \frac{2c_\Delta\alpha}{\sigma_c}(k_1^2\mathcal{H}_1 + k_2^2\mathcal{H}_2). \tag{19}$$

One can again try to construct a tracking solution of Eq. (19) by putting its left-hand side to zero. It is clear, however, that, unlike in the case (12), such a tracking solution cannot be time independent. Indeed, according to Eq. (6), $\dot{\mathcal{H}}_k = 0$ only if $\Delta\mu = \frac{2\pi k}{\alpha}$, while our solution $\Delta\mu_{\rm tr} = \frac{2\pi}{\alpha}\frac{k_1^2\mathcal{H}_1 + k_2^2\mathcal{H}_2}{k_1\mathcal{H}_1 + k_2\mathcal{H}_2}$ depends on both modes. In the case where a shorter mode (k_1) contains most of the energy density, $\Delta\mu$ will grow very fast and reach $\Delta\mu_{\rm tr} \approx \frac{2\pi k_1}{\alpha}(1-\epsilon)$. Initially, $\epsilon = \frac{k_2\mathcal{H}_2}{k_1\mathcal{H}_1} \ll 1$ as $\mathcal{H}_2$ is subdominant and $\Delta\mu$ is close to its "static" value for k_1. Therefore, the mode $\mathcal{H}_1$ remains almost constant, for $\eta < \eta_\sigma(k_1)/\epsilon(\eta)$. For the mode $\mathcal{H}_2$, however, $k_2 < \frac{\alpha\Delta\mu_{\rm tr}}{2\pi}$ and from the solution (8) [valid for any $\Delta\mu(\eta)$], we see that $\mathcal{H}_2$ will start growing. As its growth enters the exponential phase, ϵ increases and k_1 becomes greater than $\Delta\mu(\eta)$, causing $\mathcal{H}_1$ to decay exponentially. $\Delta\mu(\eta)$ will therefore quickly evolve to the value $\frac{2\pi}{\alpha}k_2$. From Eq. (6) we find that for $\epsilon \ll 1$

$$\mathcal{H}_2(\eta) \approx \mathcal{H}_2(\eta_0)e^{(2k_1k_2/\sigma_c)\eta} \tag{21}$$

and see that $\dot{\mathcal{H}}_1 = -\dot{\mathcal{H}}_2$ as long as $\Gamma_B \gg \Gamma_f$; i.e., the total helicity of the system is conserved [30].

The evolution of continuous spectra is qualitatively very similar. Assume that the initial helicity spectrum $\mathcal{H}_k^0$ has its maximum at a scale k_1 and then decays as $\mathcal{H}_k^0 \propto (\frac{k}{k_1})^{n_s-2}$, with $n_s \ge 3$. The scale k_1 determines the value of $\Delta\mu$ at the beginning, while the longer modes grow. At the moment when backreaction of these growing modes on $\Delta\mu$ becomes non-negligible, the chemical potential difference gets smaller and the modes with $k_1 \gtrsim k > \frac{\alpha\Delta\mu(\eta)}{2\pi}$ start decaying.

For discrete spectrum $\Delta\mu$ changes by "steps," defined by the modes k_n (see Fig. 1 for $\Gamma_f = 0$, red dashed curve). Every such step corresponds, in the other panel of Fig. 1, to a fast decay of one helicity mode and exponential growth of an adjacent one (from red to purple), while the total helicity remains constant (black dot-dashed line). The conservation of helicity implies that the total magnetic energy gets dissipated as $\rho_k = \frac{k}{2}\mathcal{H}_k$ for helical fields. If we sample the same spectrum with a larger number of modes, the evolution of $\Delta\mu$ becomes monotonic. If the initial spectrum is sharp ($n_s > 3$), $\Delta\mu$ will decay more slowly, and the short modes will survive for a longer time. The resulting spectrum will, however, be roughly the same—the helicity concentrates around the longest mode k_2 that had enough time to start growing, $\Delta\mu = \Delta\mu_{\rm tr}(k_2)$, and the magnetic energy is smaller by the factor $\frac{k_2}{k_1}$. We

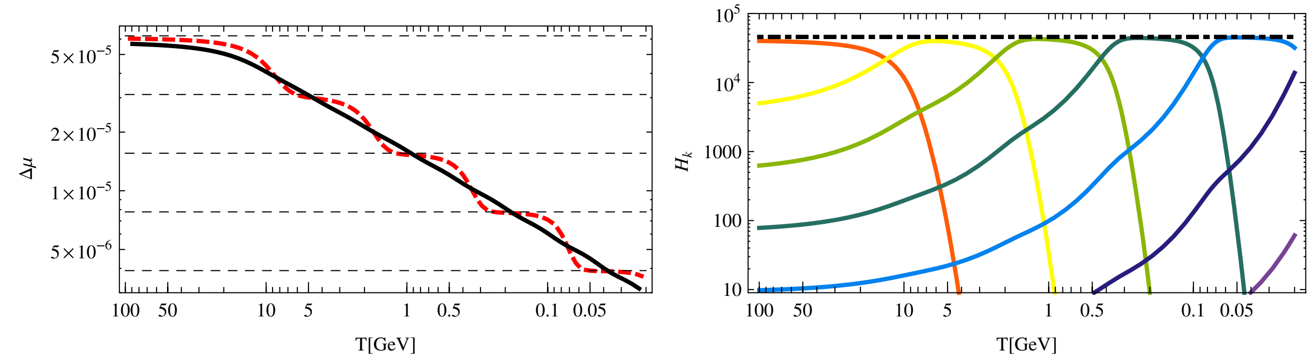

FIG. 1 (color online). Evolution in the absence of chirality flip for continuous spectrum $\mathcal{H}_k^0 \propto k^3$. Left: The evolution of $\Delta\mu(\eta)$ (black solid line). Red dashed line: The approximation of the spectrum by 10 modes ($k_n = 10^{-10} \times 2^{n-1/2}$, $n = 1, 10$). The horizontal lines are tracking solutions for individual modes k_n. Right: Transfer of helicity from the shorter to the longer modes (red to blue).

believe that these results correctly describe the interaction between different helicity modes even in the inhomogeneous case, provided that deviations from local thermodynamic equilibrium are not very dramatic and $\Delta\mu(x)$ is smooth.

Finally, the exact numerical solution of the full system with a continuous spectrum and finite $\gamma(\eta)$ is shown in Fig. 2, where the red dashed line shows $\Delta\mu(\eta)$ in the case $\Gamma_f = 0$ and the thick blue and green lines show $\Delta\mu(\eta)$ for $\Gamma_f \neq 0$ and different r_B. This full evolution follows that of $\Gamma_f = 0$ and then breaks down exponentially fast when $\gamma(\eta) \sim 1$ and (18) holds.

Conclusion.—This work demonstrates that the traditional MHD equations should be modified, when applied to a plasma of relativistic particles with $T \gg m$. The proper account of the chiral anomaly changes the evolution of magnetic fields in two ways: (i) Yhe magnetic fields survive several orders of magnitude longer than the time defined by magnetic diffusion [Eqs. (16) and (17)], and (ii) an inverse cascade develops, transferring energy from shorter to longer wavelength modes. The effect depends on the energy of magnetic fields parametrized by its ratio to the total energy density, r_B. In the literature discussing the evolution of magnetic fields (see, e.g., [20,23,27]), $r_B \leq 1$ is often considered. It was demonstrated, e.g., in Refs. [31,32] that $r_B \sim$ a few $\times 10^{-3}$ may be generated at cosmological first-order phase transitions. The mechanism of Ref. [33] predicts maximally helical magnetic fields with $B \sim 100$ GeV^2 (i.e., $r_B \sim 10^{-2}$; cf. [34]) at small scales. See [6,31–33,35,36], and references therein.

We refer to the value of r_B at $T \sim 100$ GeV. Because of the inverse cascade and helicity conservation, this energy decreases by about an order of magnitude by the time when our effect stops ($T \sim$ 10–100 MeV). The subsequent evolution of the magnetic fields is described by the conventional MHD [5,20,22]; a significant part of r_B further dissipates, so that only the large scale tail of the spectrum may survive due to turbulent effects. To predict the final fate of these CMFs for every initial spectrum and compare them with cosmological bounds (see, e.g., [37]), our results should be combined with the MHD analysis. Nevertheless, the above-described mechanism, based entirely on the standard model, clearly improves the chances of survival of CMFs generated at subhorizon scales [38]. Indeed, even for $r_B \sim 10^{-5}$ the fields survive down to $T \sim 100$ MeV, while for $r_B \sim 0.1$ the inverse cascade is operational down to $T \sim 10$ MeV. Moreover, regardless of the survival of the CMFs, this effect is important as the left-right asymmetry in the electron sector survives down to $T \sim \mathcal{O}(100)$ MeV and thus potentially affects important processes in the early Universe: It can change the nature of the QCD phase transition [39] and produce gravitational waves [40], which would leave its imprints on big bang nucleosynthesis and the cosmic microwave background [41,42].

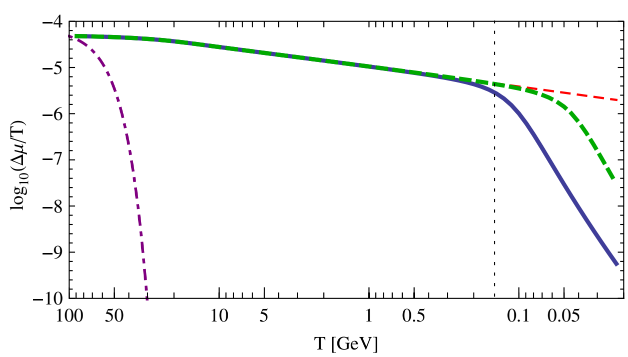

FIG. 2 (color online). Evolution of the chemical potential for $r_B = 5 \times 10^{-5}$ (solid blue line). The vertical line at $T \approx 150$ MeV marks $\gamma = 1$, $\Delta\mu(150\text{ MeV}) \simeq 2.9 \times 10^{-6}$. The red dashed line shows evolution of the chemical potential for $\Gamma_f = 0$. The green short-dashed line shows $\Delta\mu(\eta)$ for $r_B = 5 \times 10^{-4}$. The purple dash-dotted line shows the decay of the chemical potential in the absence of a magnetic field.

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