Quantum Gravitational Contributions to the Cosmic Microwave Background Anisotropy Spectrum

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We derive the primordial power spectrum of density fluctuations in the framework of quantum cosmology. For this purpose we perform a Born-Oppenheimer approximation to the Wheeler-DeWitt equation for an inflationary universe with a scalar field. In this way, we first recover the scale-invariant power spectrum that is found as an approximation in the simplest inflationary models. We then obtain quantum gravitational corrections to this spectrum and discuss whether they lead to measurable signatures in the cosmic microwave background anisotropy spectrum. The nonobservation so far of such corrections translates into an upper bound on the energy scale of inflation.

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Without observational guidance it is illusory to find the correct quantum theory of gravity. While there exist various approaches to such a theory (see, e.g., the overview in [1]), definite predictions are rare. Among these are the calculation of small quantum gravitational corrections to the Newtonian and Coulomb potential [2] and corrections to Lamb shift and other effects due to the possible existence of a minimal length [3]. While the first effects are too tiny to be observable in the foreseeable future, the latter depend on a new dimensionless parameter for which bounds can be found. The calculation of effects is also important for the comparison of different approaches and for the decision whether gravity must be quantized at all.

Our purpose here is to calculate potential observational contributions to the cosmic microwave background (CMB) anisotropy spectrum from quantum gravity. After all, the main applications of such a theory should arise from cosmology and black-hole physics. Our framework will be quantum geometrodynamics governed by the Wheeler-DeWitt equation [1,4]. Although it is likely that this is not the most fundamental approach, one can put forward strong arguments that it is approximately valid at energy scales somewhat smaller than the Planck mass [5]. For example, if one looks for a quantum wave equation that immediately leads to Einstein's equations in the semiclassical limit, one is directly driven to the Wheeler-DeWitt equation.

This connection between the Wheeler-DeWitt equation and quantum field theory in an external spacetime can be established by a Born-Oppenheimer type of approximation [1]. Expanding with respect to the Planck mass, one arrives first at the functional Schrödinger equation for nongravitational fields in an external spacetime satisfying the Einstein equations. Proceeding with this scheme to the next orders, one can derive quantum gravitational correction terms proportional to the inverse Planck mass squared. The dominating correction terms are calculated in [6] at the formal level of the full equations. The complete set of correction terms at this order together with their interpretation in terms of Feynman diagrams can be found in [7]. The generalization of [6] to supergravity is presented in [8].

In the present Letter, we calculate the dominating correction term of [6] for the case of the CMB anisotropy spectrum. In this way, we hope that either a quantum gravitational effect can be observed or that bounds on cosmological parameters can be found from their nonobservation.

We shall consider the Wheeler-DeWitt equation for the case of small fluctuations (leading to the anisotropies in the CMB spectrum) in a flat Friedmann-Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and a scalar-field ϕ that plays the role of the inflaton. For definiteness we shall choose the simplest potential in chaotic inflation [9], $\mathcal{V}(\phi) = \frac{1}{2}m^2\phi^2$, but any other potential should fit our purpose as long as at the classical level a slow-roll condition of the form $\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$ holds. Setting $\hbar = c = 1$, the Wheeler—DeWitt equation for this "minisuperspace part" reads (see, e.g., [1])

$$\mathcal{H}_0\Psi_0(\alpha,\phi) \equiv \frac{e^{-3\alpha}}{2}\left[\frac{1}{m_{\rm P}^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}m^2\phi^2\right]\Psi_0(\alpha,\phi) = 0, \quad (1)$$

where $m_{\rm P} = \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}$ GeV is a rescaled Planck mass, and the field redefinition $\phi \rightarrow \phi/\sqrt{2}\pi$ was performed.

In addition to the Born-Oppenheimer approximation for the Wheeler-DeWitt equation, we make one further assumption: we assume that the kinetic term of the ϕ field is small compared to the potential term, that is, $\partial^2\Psi_0/\partial\phi^2 \ll e^{6\alpha}m^2\phi^2\Psi_0$. It corresponds to the slow-roll approximation for inflationary models and is also the standard assumption in discussions of the no-boundary and tunneling proposals in quantum cosmology [1]; it allows us to neglect the ϕ-kinetic term in (1). For this reason we can also substitute $m\phi$ in (1) by $m_{\rm P}H$, where H is the quasi-static Hubble parameter of inflation, which in the classical limit obeys $|\dot{H}| \ll H^2$. This replacement of the quantum variable ϕ by a c number is not problematic here, because (1) describes in the Born-Oppenheimer approximation the classical background on which the quantum fluctuations of the inflaton propagate, see below.

We now consider the fluctuations of an inhomogeneous inflaton field on top of its homogeneous part,

$$\phi \to \phi(t) + \delta\phi(\mathbf{x}, t),$$

and perform a decomposition into Fourier modes with wave vector $\mathbf{k}$, $k \equiv |\mathbf{k}|$,

$$\delta\phi(\mathbf{x}, t) = \sum_k f_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

(We assume for simplicity that space is compact and the spectrum for $\mathbf{k}$ thus discrete.) The Wheeler-DeWitt equation including the fluctuation modes then reads [10]

$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right]\Psi(\alpha, \phi, \{f_k\}_{k=1}^{\infty}) = 0,$$

where the Hamiltonians $\mathcal{H}_k$ of the fluctuation modes are given by

$$\mathcal{H}_k = \frac{1}{2} e^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + (k^2 e^{4\alpha} + m^2 e^{6\alpha}) f_k^2\right].$$

Since the fluctuations are small, their self-interaction can be neglected, and one can make the following product ansatz for the full wave function:

$$\Psi(\alpha, \phi, \{f_k\}_{k=1}^{\infty}) = \Psi_0(\alpha, \phi) \prod_{k=1}^{\infty} \tilde{\Psi}_k(\alpha, \phi, f_k).$$

Under some mild assumptions, one finds that the components $\Psi_k(\alpha, \phi, f_k) := \Psi_0(\alpha, \phi)\tilde{\Psi}_k(\alpha, \phi, f_k)$ obey [10,11]

$$\frac{1}{2} e^{-3\alpha}\left[\frac{1}{m_{\rm P}^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha} m_{\rm P}^2 H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2\right]\Psi_k(\alpha, \phi, f_k) = 0, \qquad (2)$$

where we have defined the quantity

$$W_k(\alpha) := k^2 e^{4\alpha} + m^2 e^{6\alpha},$$

and we have used $m\phi \approx m_{\rm P}H$ as mentioned above. (For this reason we shall omit the argument ϕ in the following.) Eq. (2) is the starting point for the Born-Oppenheimer approximation.

Following the general procedure of [6], we make the ansatz

$$\Psi_k(\alpha, f_k) = e^{iS(\alpha, f_k)},$$

and expand $S(\alpha, f_k)$ in terms of powers of $m_{\rm P}^2$,

$$S(\alpha, f_k) = m_{\rm P}^2 S_0 + m_{\rm P}^0 S_1 + m_{\rm P}^{-2} S_2 + \ldots.$$

Inserting this ansatz into (2) and comparing consecutive orders of $m_{\rm P}^2$, one obtains at $\mathcal{O}(m_{\rm P}^4)$ that S_0 is independent of f_k, and that it obeys at $\mathcal{O}(m_{\rm P}^2)$ the Hamilton-Jacobi equation

$$\left[\frac{\partial S_0}{\partial\alpha}\right]^2 - V(\alpha) = 0, \qquad V(\alpha) := e^{6\alpha} H^2,$$

which defines the classical minisuperspace background. Its solution is $S_0(\alpha) = \pm e^{3\alpha} H/3$.

At $\mathcal{O}(m_{\rm P}^0)$ we first write

$$\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha) e^{iS_1(\alpha, f_k)},$$

and impose a condition on $\gamma(\alpha)$ that makes it equal to the standard WKB prefactor. After introducing the "WKB time" according to

$$\frac{\partial}{\partial t} := -e^{-3\alpha}\frac{\partial S_0}{\partial\alpha}\frac{\partial}{\partial\alpha}, \qquad (3)$$

one finds that each $\psi_k^{(0)}$ obeys a Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}. \qquad (4)$$

At the next order $\mathcal{O}(m_{\rm P}^{-2})$, we decompose $S_2(\alpha, f_k)$ as follows:

$$S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k),$$

and demand that $\varsigma(\alpha)$ be the standard second-order WKB correction. The wave functions

$$\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k) e^{im_{\rm P}^{-2}\eta(\alpha, f_k)},$$

then obey the quantum gravitationally corrected Schrödinger equation [6]

$$i\frac{\partial}{\partial t}\psi_k^{(1)} = \mathcal{H}_k\psi_k^{(1)} - \frac{e^{3\alpha}}{2m_{\rm P}^2\psi_k^{(0)}}\left[\frac{(\mathcal{H}_k)^2}{V}\psi_k^{(0)} + i\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_k}{V}\right)\psi_k^{(0)}\right]\psi_k^{(1)}. \qquad (5)$$

In the following, we shall only take into account the first correction term because it usually gives the dominating contribution [6,7]. The second correction term corresponds to a small violation of unitarity, where unitarity is here understood with respect to the standard $\mathcal{L}^2$ inner product for the modes f_k. While the Hilbert-space structure for full quantum gravity is unknown [1], this is the obvious choice for the f_k because their states ψ_k obey the approximate Schrödinger equation (4). The unitarity-violating term can be absorbed in a t-dependent redefinition of the states [12].

We shall now look for a solution of the uncorrected Schrödinger equation (4). We make a Gaussian ansatz,

$$\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t) e^{-(1/2)\Omega_k^{(0)}(t) f_k^2}. \qquad (6)$$

Here, we have expressed α in terms of the WKB time t introduced in (3), $\alpha = Ht$. We thereby arrive at the following system of differential equations:

$$\dot{\mathcal{N}}_k^{(0)}(t) = -\frac{i}{2}e^{-3\alpha}\mathcal{N}_k^{(0)}(t)\Omega_k^{(0)}(t), \tag{7}$$

$$\dot{\Omega}_k^{(0)}(t) = ie^{-3\alpha}[-\{\Omega_k^{(0)}(t)\}^2 + W_k(t)]. \tag{8}$$

In the model of chaotic inflation employed here we have the condition $(m/H)^2 \ll 1$ [9]. In this limit the solution of (8) expressed in terms of the dimensionless quantity $\xi(t) := k/(Ha(t))$ reads

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2\xi}\frac{1}{\xi - i} + \mathcal{O}\left(\frac{m^2}{H^2}\right). \tag{9}$$

From (7) and the normalization of the states one then obtains the solution $|\mathcal{N}_k^{(0)}(t)|^2 = [\Re\mathrm{e}\Omega_k^{(0)}(t)/\pi]^{1/2}$.

In the slow-roll regime, the density contrast is given by (see, e.g., [13], p. 364)

$$\delta_k(t) \approx \frac{\delta\rho_k(t)}{\mathcal{V}_0} = \frac{\dot{\phi}(t)\dot{\sigma}_k(t)}{\mathcal{V}_0},$$

where $\mathcal{V}_0$ denotes the scalar-field potential evaluated at the background solution $\phi(t)$, and $\sigma_k(t)$ is the classical quantity related to the quantum mechanical variable $f_k(t)$ by taking its expectation value with respect to a Gaussian state; for a general Gaussian we define

$$\begin{aligned}\sigma_k^2(t) &:= \langle\psi_k|f_k^2|\psi_k\rangle \\ &= \sqrt{\frac{\Re\mathrm{e}\Omega_k}{\pi}}\int_{-\infty}^{\infty} f_k^2 e^{-(1/2)[\Omega_k^*(t)+\Omega_k(t)]f_k^2}df_k \\ &= \frac{1}{2\,\Re\mathrm{e}\Omega_k(t)}.\end{aligned}$$

The density contrast must be evaluated at the time t_{enter} when the corresponding mode reenters the Hubble radius during the radiation-dominated phase. A standard relation gives ([13], p. 367)

$$\delta_k(t_{\mathrm{enter}}) = \frac{4}{3}\frac{\mathcal{V}_0}{\dot{\phi}^2}\delta_k(t_{\mathrm{exit}}) = \frac{4}{3}\frac{\dot{\sigma}_k(t)}{\dot{\phi}(t)}\bigg|_{t=t_{\mathrm{exit}}}.$$

Evaluating $\dot{\sigma}_k^{(0)}(t)$ at $t = t_{\mathrm{exit}}$ using (9) and noting that $\xi(t_{\mathrm{exit}}) = 2\pi$ at Hubble-scale crossing, we get

$$|\dot{\sigma}_k^{(0)}(t)|_{t=t_{\mathrm{exit}}} = \frac{2\sqrt{2}\pi^2}{\sqrt{4\pi^2+1}}\frac{H^2}{k^{3/2}}.$$

This then leads to the power spectrum

$$\Delta_{(0)}^2(k) := 4\pi k^3|\delta_k(t_{\mathrm{enter}})|^2 \propto \frac{H^4}{|\dot{\phi}(t)|^2_{t_{\mathrm{exit}}}}, \tag{10}$$

which is approximately scale invariant. This is the standard result for a generic inflationary model.

We now want to calculate the quantum gravitational correction terms following from (5). (A possible effect on the relic graviton density is discussed along these lines in [14].) As mentioned above, we shall neglect the unitarity-violating term in (5). We assume that the correction can be accommodated by the Gaussian ansatz

$$\psi_k^{(1)}(t, f_k) = \left(\mathcal{N}_k^{(0)}(t) + \frac{1}{m_{\mathrm{P}}^2}\mathcal{N}_k^{(1)}(t)\right) \times \exp\left[-\frac{1}{2}\left(\Omega_k^{(0)}(t) + \frac{1}{m_{\mathrm{P}}^2}\Omega_k^{(1)}(t)\right)f_k^2\right].$$

One then gets from (5) an equation for the correction term $\Omega_k^{(1)}$,

$$\dot{\Omega}_k^{(1)}(t) \approx -2ie^{-3\alpha}\Omega_k^{(0)}(t) \times\left(\Omega_k^{(1)}(t) - \frac{3}{4V(t)}\left[[\Omega_k^{(0)}(t)]^2 - W_k(t)\right]\right). \tag{11}$$

We shall assume that the correction term vanishes for late times, $\Omega_k^{(1)}(t) \to 0$ as $t \to \infty$. This is, of course, an assumption that must eventually be justified from the theory itself; the chosen boundary condition guarantees that the model is consistent and in accordance with observations at late times.

Using (9), we rewrite (11) in terms of ξ, which in the limit $(m/H)^2 \ll 1$ gives

$$\frac{d}{d\xi}\Omega_k^{(1)}(\xi) = \frac{2i\xi}{\xi - i}\Omega_k^{(1)}(\xi) + \frac{3\xi^3}{2}\frac{2\xi - i}{(\xi - i)^3}. \tag{12}$$

The corrected quantity $\dot{\sigma}_k^{(1)}$ needed for the evaluation of the power spectrum (10) is then given by

$$\begin{aligned}|\dot{\sigma}_k^{(1)}(t)| &= \left|\frac{H\xi}{\sqrt{2}}\frac{d}{d\xi}\left[\left(\Re\mathrm{e}\Omega_k^{(0)}(\xi) + \frac{1}{m_{\mathrm{P}}^2}\Re\mathrm{e}\Omega_k^{(1)}(\xi)\right)^{-1/2}\right]\right| \\ &= \left|\frac{\xi^2}{\sqrt{2(\xi^2+1)}}\frac{H^2}{k^{3/2}}\left(1 + \frac{\xi^2+1}{k^3}\Re\mathrm{e}\Omega_k^{(1)}(\xi)\frac{H^2}{m_{\mathrm{P}}^2}\right)^{-3/2}\left(1 - \frac{(\xi^2+1)^2}{2\xi k^3}\Re\mathrm{e}\left[\frac{d}{d\xi}\Omega_k^{(1)}(\xi)\right]\frac{H^2}{m_{\mathrm{P}}^2}\right)\right|.\end{aligned}$$

The solution of (12) can be reduced to numerical integration and yields $\Re\mathrm{e}\Omega_k^{(1)}(\xi = 2\pi) \simeq -1.076$ as well as $\Re\mathrm{e}[d\Omega_k^{(1)}(\xi)/d\xi]_{\xi=2\pi} \simeq 1.451$, which eventually leads to

$$|\dot{\sigma}_k^{(1)}|_{t_{\mathrm{exit}}} \simeq |C_k||\dot{\sigma}_k^{(0)}|_{t_{\mathrm{exit}}}, \tag{13}$$

where

$$C_k := \left(1 - \frac{43.56}{k^3}\frac{H^2}{m_{\mathrm{P}}^2}\right)^{-(3/2)}\left(1 - \frac{189.18}{k^3}\frac{H^2}{m_{\mathrm{P}}^2}\right). \tag{14}$$

With this result we can write the corrected power spectrum as the product of the uncorrected power spectrum with a correction term C_k, $\Delta_{(1)}^2(k) = \Delta_{(0)}^2(k)C_k^2$. An expansion of C_k^2 in terms of $(H/m_{\mathrm{P}})^2$ yields

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k)\left[1 - \frac{123.83}{k^3}\frac{H^2}{m_P^2} + \frac{1}{k^6}\mathcal{O}\left(\frac{H^4}{m_P^4}\right)\right]^2. \quad (15)$$

We emphasize the important fact that the corrected power spectrum is now explicitly scale dependent. The quantitative contribution of the quantum gravitational terms is only significant if the inflationary Hubble parameter H is sufficiently large. It is not surprising that the effects become sizeable only if H approaches the Planck scale.

An inspection of (14) shows that C_k approaches one for large k (as it must), but decreases monotonically to zero for large scales (small k); one thus finds a *suppression of power* for large scales. The zero point is reached for $k \approx 5.74(H/m_P)^{2/3}$. However, the approximation (14) breaks down if this zero point is approached and one has to take into account in this limit higher orders of $(H/m_P)^2$.

The effect is most prominent for large scales because these scales are the earliest to leave the Hubble scale during inflation. However, the measurement accuracy for large scales is fundamentally limited by cosmic variance, which follows from the fact that we only observe one Universe (see, e.g., [15]). For this reason, missions such as the PLANCK satellite will not be able to see this effect if it has not already been seen now. But there is still a merit of our analysis: from the current nonobservation of the quantum gravity terms one can get an upper bound on the inflationary Hubble scale. Assuming for a rough estimate that C_k^2 is not less than around 0.95 for the largest observable scales $k \sim 1$ (which is motivated by the fact that the deviation of the observed power spectrum from a scale-invariant spectrum is smaller than about 5% [16]), one obtains from (15) the bound

$$H \lesssim 1.4 \times 10^{-2} m_P \sim 4 \times 10^{17}\ \text{GeV}. \quad (16)$$

We must emphasize, however, that there already exists a stronger constraint on this scale. This is because the energy scale of inflation is limited by the observational bound on the tensor-to-scalar ratio r (see, e.g., [17]). Using $r < 0.22$ [16] one finds $H \lesssim 10^{-5} m_P \sim 10^{14}$ GeV. As emphasized, for example, in [18], the assumption $H \ll m_P$ is required anyway and is self-consistent for inflationary models to have a connection with reality. For the limiting value $H \sim 10^{14}$ GeV one gets from (15)

$$\Delta^2_{(1)}(k) \simeq \Delta^2_{(0)}(k)\left[1 - 1.76 \times 10^{-9}\frac{1}{k^3} + \frac{\mathcal{O}(10^{-15})}{k^6}\right]^2;$$

for this value the correction is, thus, too small to be seen in present observations.

In spite of this, we emphasize that our constraint (16) arises as a definite prediction from a conservative approach to quantum gravity, and it is reassuring that it is consistent with other limits. It indicates, in particular, that no additional trans-Planckian effects (see, e.g., [15,18]) have to be taken into account in order to understand the predictions of this model.

Quantum gravitational corrections to the CMB anisotropy spectrum have also been derived in loop quantum cosmology. While in [19], a suppression of power at large scales was found, the authors of [20] predicted an enhancement at those scales. This demonstrates that one can use the CMB anisotropies to compare different approaches to quantum gravity. We hope that such investigations will eventually lead to an observational test of quantum gravity.

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