# Expanding (3 + 1)-Dimensional Universe from a Lorentzian Matrix Model for Superstring Theory in (9 + 1) Dimensions

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We reconsider the matrix model formulation of type IIB superstring theory in (9 + 1)-dimensional space-time. Unlike the previous works in which the Wick rotation was used to make the model well defined, we regularize the Lorentzian model by introducing infrared cutoffs in both the spatial and temporal directions. Monte Carlo studies reveal that the two cutoffs can be removed in the large-$N$ limit and that the theory thus obtained has no parameters other than one scale parameter. Moreover, we find that three out of nine spatial directions start to expand at some "critical time," after which the space has SO(3) symmetry instead of SO(9).

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*Introduction.*—One of the most fundamental questions concerning our Universe is why we live in a (3 + 1)-dimensional space-time, and why the Universe is expanding. The aim of this Letter is to provide some evidence that these facts can be derived from a nonperturbative formulation of superstring theory in (9 + 1) dimensions based on matrix models. Motivated by recent developments in understanding the dynamics of the Euclideanized model, we study the SO(9,1) symmetric Lorentzian model nonperturbatively without Wick rotation. Our Monte Carlo results demonstrate, among other things, that three out of nine spatial directions start to expand in the early Universe. We expect that what we are doing here is essentially a first-principle calculation of the unified theory including quantum gravity. This may be contrasted with the quantum cosmology in the early 1980s that aimed at describing the birth of the Universe [1] within the minisuperspace approximation. More recently, a nonperturbative approach to quantum gravity has been pursued using the causal dynamical triangulation [2]. For earlier works that put forward the idea to use matrices for cosmology, see Refs. [3,4].

*Matrix model for superstrings.*—Superstring theory not only provides a most natural candidate for a consistent theory of quantum gravity but also enables a unified description of all the interactions and the matters. A crucial problem is that we do not yet have a well-established nonperturbative formulation, which would be needed in addressing dynamical issues such as the determination of space-time dimensionality [5].

In the 1990s, there was remarkable progress in understanding the nonperturbative aspects of superstring theory based on $D$-branes. Most importantly, it was noticed that large-$N$ matrices are the appropriate microscopic degrees of freedom which are useful in formulating superstring theory in a nonperturbative manner [6–8]. In particular, the type IIB matrix model was proposed as a nonperturbative formulation of type IIB superstring theory in ten-dimensional space-time [7]. It was also realized that the five types of superstring theory in ten dimensions are just different descriptions of the same theory. Therefore, it was speculated that the type IIB matrix model actually describes the unique underlying theory, although it takes the form that has explicit connection to perturbative type IIB superstring theory.

In the type IIB matrix model, the space-time is represented *dynamically* by the eigenvalue distribution of ten bosonic $N \times N$ traceless Hermitian matrices [9]. So far, the dynamical generation of four-dimensional space-time has been discussed exclusively in the Euclideanized model. Indeed the spontaneous symmetry breaking (SSB) of SO(10) down to SO(4) was suggested by the Gaussian expansion method [10,11]. Recently, systematic calculation of the free energy has been performed for SO($d$) symmetric vacua with $2 \le d \le 7$, and it is found that $d = 3$ gives the minimum [12]. Furthermore, the ratio of the space-time extent in the extended directions to that in the shrunken directions is shown to be finite. These results, if true, suggest the necessity for reconsidering the formulation in order to make any connection to the real world.

*Matrix model with SO(9,1) symmetry.*—Our starting point is the action $S = S_b + S_f$, where [7]

$$S_b = -\frac{1}{4g^2}\,\mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu]),$$
$$S_f = -\frac{1}{2g^2}\,\mathrm{tr}(\Psi_\alpha(\mathcal{C}\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta]), \tag{1}$$

 

with $A_\mu$ ($\mu = 0, \ldots, 9$) and $\Psi_\alpha$ ($\alpha = 1, \ldots, 16$) being $N \times N$ traceless Hermitian matrices. The Lorentz indices $\mu$ and $\nu$ are contracted using the metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. The $16 \times 16$ matrices $\Gamma^\mu$ are ten-dimensional gamma matrices after the Weyl projection, and the unitary matrix $\mathcal{C}$ is the charge conjugation matrix. The action has manifest SO(9,1) symmetry, where $A_\mu$ and $\Psi_\alpha$ transform as a vector and a Majorana-Weyl spinor, respectively. The Euclidean model, which has SO(10) symmetry, can be obtained from this action by the Wick rotation $A_0 = iA_{10}$. A crucial difference is that the bosonic part of the action in the Euclidean model is positive definite, whereas in the Lorentzian model it is

$$\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -2\mathrm{tr}(F_{0i})^2 + \mathrm{tr}(F_{ij})^2, \tag{2}$$

where $F_{\mu\nu} = -i[A_\mu, A_\nu]$ are Hermitian matrices, and hence the two terms in (2) have opposite signs [13].

We study, for the first time, the Lorentzian model non-perturbatively based on the partition function

$$Z = \int dA d\Psi e^{iS} = \int dA e^{iS_b} \mathrm{Pf}\,\mathcal{M}(A), \tag{3}$$

where the Pfaffian $\mathrm{Pf}\,\mathcal{M}(A)$ appears from integrating out the fermionic matrices $\Psi_\alpha$. Note that in the Euclidean model, the Pfaffian is complex in general, and its phase plays a crucial role in the aforementioned SSB of SO(10) symmetry [14,15]. On the other hand, the Pfaffian in the Lorentzian model is *real*. Therefore, the mechanism of SSB that was identified in the Euclidean model is absent in the Lorentzian model.

In the definition (3), we have replaced the "Boltzmann weight" $e^{-S}$ used in the Euclidean model by $e^{iS}$. This is theoretically motivated from the connection to the world sheet theory [7]. The partition function (3) can also be obtained formally from pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in $(9 + 1)$ dimensions by dimensional reduction. Note, however, that the expression (3) is ill defined and requires appropriate regularization in order to make any sense out of it. This is in striking contrast to the Euclidean model, in which the partition function is shown to be finite without any regularization [16,17].

It turns out that the integration over $A_0$ is divergent, even if we fix $\frac{1}{N}\mathrm{tr}(A_i)^2$ to a constant. In order to cure this divergence, we introduce a constraint

$$\frac{1}{N}\mathrm{tr}(A_0)^2 \le \kappa \frac{1}{N}\mathrm{tr}(A_i)^2, \tag{4}$$

which is invariant under the scale transformation $A_\mu \to \rho A_\mu$. Note that this constraint generically breaks SO(9,1) symmetry down to SO(9). However, it turns out to be equivalent to imposing (4) after "gauge fixing" the boost symmetry by requiring that $\frac{1}{N}\mathrm{tr}(\tilde{A}_0)^2$ with $\tilde{A}_\mu = O_{\mu\nu}A_\nu$ be minimized with respect to $O \in \mathrm{SO}(9,1)$. In this sense, the constraint actually respects the SO(9,1) symmetry.

Let us note that $e^{iS_b}$ in the partition function (3) is a phase factor just as in the path-integral formulation of quantum field theories in Minkowski space. As is commonly done in integrating oscillating functions, we introduce the convergence factor $e^{-\epsilon|S_b|}$ and take the $\epsilon \to 0$ limit after the integration.

The partition function can then be rewritten as

$$Z = \int dA \int_0^\infty dr \delta\Big(\frac{1}{N}\mathrm{tr}(A_i)^2 - r\Big) e^{iS_b - \epsilon|S_b|}\mathrm{Pf}\,\mathcal{M},$$

where the integration over $A_\mu$ is assumed to be restricted by the constraint (4). After rescaling the variables $A_\mu \to r^{1/2}A_\mu$ in the integrand, we integrate over $r$ and get

$$\int_0^\infty dr r^{18(N^2-1)/2-1} e^{r^2(iS_b - \epsilon|S_b|)} \propto \frac{1}{|S_b|^{18(N^2-1)/4}}, \tag{5}$$

which diverges for $S_b = 0$. In order to cure this divergence, we introduce a constraint

$$\frac{1}{N}\mathrm{tr}(A_i)^2 \le L^2 \tag{6}$$

before the rescaling. Then the integration domain for $r$ becomes $[0, L^2]$, and (5) is replaced by $f(S_b)$, where $f(x)$ is a function with a sharp peak at $x = 0$. Thus we arrive at the model

$$Z = \int dA f\Big(\frac{1}{N}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - C\Big)\mathrm{Pf}\,\mathcal{M}(A) \times \delta\Big(\frac{1}{N}\mathrm{tr}(A_i)^2 - 1\Big)\theta\Big(\kappa - \frac{1}{N}\mathrm{tr}(A_0)^2\Big), \tag{7}$$

where $\theta(x)$ is the Heaviside step function. The constant $C$ should be set to zero according to our derivation. If we consider the $C < 0$ case, the model (7) may be viewed as the matrix model motivated from the space-time uncertainty principle [18] with the regularization in the second line, which we find to be necessary. Since the Pfaffian $\mathrm{Pf}\,\mathcal{M}(A)$ is real in the present Lorentzian case, the model (7) can be studied by Monte Carlo simulation without the sign problem. Note that this is usually not the case for quantum field theories in Minkowski space.

*Monte Carlo results.*—We perform Monte Carlo simulation of the model (7) with $C = 0$ by using the rational hybrid Monte Carlo algorithm [19], which is quite standard in recent simulations of quantum chromodynamics including the effects of dynamical quarks.

In order to see the time evolution, we diagonalize $A_0$, and define the eigenvectors $|t_a\rangle$ corresponding to the eigenvalues $t_a$ of $A_0$ ($a = 1, \ldots, N$) with the specific order $t_1 < \cdots < t_N$. The spatial matrix in this basis $\langle t_a|A_i|t_b\rangle$ is not diagonal, but it turns out that the off-diagonal elements decrease rapidly as one goes away from a diagonal element. This motivates us to define $n \times n$ matrices

$\bar{A}_i^{(ab)}(t) \equiv \langle t_{\nu+a}|A_i|t_{\nu+b}\rangle$ with $1 \leq a,\ b \leq n$ and $t = \frac{1}{n}\sum_{a=1}^{n} t_{\nu+a}$ for $\nu = 0, \ldots, (N-n)$. These matrices represent the 9$d$ space structure at fixed time $t$ [20]. The block size $n$ should be large enough to include non-negligible off-diagonal elements. In Fig. 1 we plot the extent of space $R(t)^2 \equiv \frac{1}{n}\,\mathrm{tr}\,\bar{A}_i(t)^2$ for $N = 16$ and $n = 4$. Since the result is symmetric under the time reflection $t \to -t$ as a consequence of the symmetry $A_0 \to -A_0$, we only show the results for $t < 0$. There is a critical $\kappa$, beyond which the peak at $t = 0$ starts to grow.

Next we study the spontaneous breaking of the SO(9) symmetry. As an order parameter, we define the $9 \times 9$ (positive definite) real symmetric tensor

$$T_{ij}(t) = \frac{1}{n}\,\mathrm{tr}\{\bar{A}_i(t)\bar{A}_j(t)\}, \tag{8}$$

which is an analog of the moment of inertia tensor. The nine eigenvalues of $T_{ij}(t)$ are plotted against $t$ in Fig. 2 for $\kappa = 4.0$. We find that three largest eigenvalues of $T_{ij}(t)$ start to grow at the critical time $t_c$, which suggests that the SO(9) symmetry is spontaneously broken down to SO(3) after $t_c$. Note that $R(t)^2$ is given by the sum of nine eigenvalues of $T_{ij}(t)$.

*Mechanism of the SSB.*—The SSB of SO(9) looks mysterious at first sight, but we can actually understand the mechanism quite intuitively. Let us consider the case in which $\kappa$ is large. Then the first term of (2) becomes a large negative value, and therefore the second term has to become large in order to make (2) close to zero as required in (7). Because of the constraint $\frac{1}{N}\,\mathrm{tr}(A_i)^2 = 1$, however, it is more efficient to maximize the second term of (2) at some fixed time. The system actually chooses the middle point $t = 0$, where the suppression on $A_i$ from the first term of (2) becomes the least. This explains why the peak of $R(t)$ at $t = 0$ grows as we increase $\kappa$.

Let us then consider a simplified question: what is the configuration of $A_i$ which gives the maximum $\frac{1}{N}\,\mathrm{tr}(F_{ij})^2$ with fixed $\frac{1}{N}\,\mathrm{tr}(A_i)^2 = 1$. Using the Lagrange multiplier $\lambda$, we maximize the function $G = \mathrm{tr}(F_{ij})^2 - \lambda\,\mathrm{tr}(A_i)^2$. Taking the derivative with respect to $A_i$, we obtain $2[A_j, [A_j, A_i]] - \lambda A_i = 0$. This equation can be solved if $A_i = \chi L_i$ for $i \leq d$, and $A_i = 0$ for $d < i \leq 9$, where $L_i$ are the representation matrices of a compact semisimple Lie algebra with $d$ generators. Clearly $d$ should be less than or equal to 9. It turns out that the maximum of $\frac{1}{N}\,\mathrm{tr}(F_{ij})^2$ is achieved for the SU(2) algebra, which has $d = 3$, with $L_i$ being the direct sum of the spin-$\frac{1}{2}$ representation and $(N-2)$ copies of the trivial representation. This implies the SSB of SO(9) down to SO(3). The SSB can thus be understood as a classical effect in the $\kappa \to \infty$ limit. When we tune $\kappa$ with increasing $N$ as described below, quantum effects become important. We have confirmed [21] that the $n \times n$ matrix $Q = \sum_{i=1}^{9} \bar{A}_i(t)^2$ has quite a continuous eigenvalue distribution, which implies that the space is not like a two-dimensional sphere as one might suspect from the classical picture.

*Removing the cutoffs.*—It turned out that one can remove the infrared cutoffs $\kappa$ and $L$ in the large-$N$ limit in such a way that $R(t)$ scales. This can be done in two steps. (i) First we send $\kappa$ to $\infty$ with $N$ as $\kappa = \beta N^p$ ($p \simeq \frac{1}{4}$) [21]. The scaling curve of $R(t)$ one obtains in this way depends on $\beta$. (ii) Next we send $\beta$ to $\infty$ with $L$. The two limits correspond to the continuum limit and the infinite volume limit, respectively, in quantum field theory. Thus the two constraints (4) and (6) can be removed in the large-$N$ limit, and the resulting theory has no parameter other than one scale parameter.

Let us discuss the second limit (ii) in more detail. We find that the inequality (6) is actually saturated for the dominant configurations. Therefore, one only has to

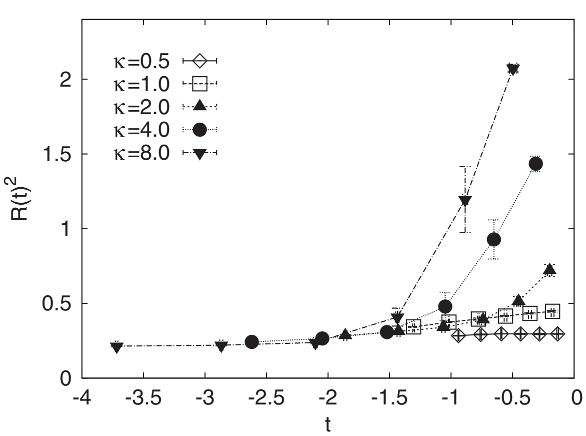


FIG. 1. The extent of space $R(t)^2$ with $N = 16$ and $n = 4$ is plotted as a function of $t$ for five values of $\kappa$. The peak at $t = 0$ starts to grow at some critical $\kappa$.

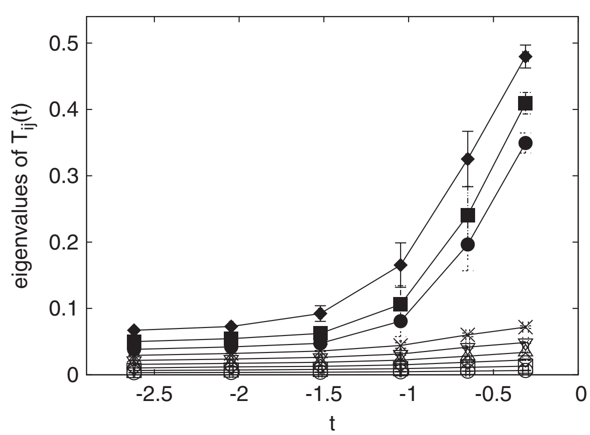


FIG. 2. The nine eigenvalues of $T_{ij}(t)$ with $N = 16$ and $n = 4$ are plotted as a function of $t$ for $\kappa = 4.0$. After the critical time $t_c$, three eigenvalues become larger, suggesting that the SO(9) symmetry is spontaneously broken down to SO(3).

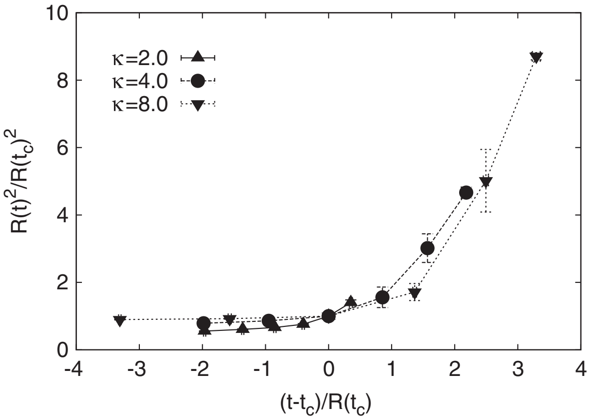


FIG. 3. Our data for $R(t)^2$ shown in Fig. 1 with $\kappa$ larger than the critical value are replotted against the shifted time $t - t_c$ in units of the size of the Universe $R(t_c)$ at the critical time.

make the rescaling $A_\mu \mapsto LA_\mu$ in order to translate the configurations in the model (7) as those in the original partition function. It turns out that $R(t)$ for the rescaled configurations scales in $\beta$ by tuning $L$ and shifting $t$ appropriately. In order to see this, it is convenient to choose $L$ so that $R(t)$ at the critical time $t = t_c$ becomes unity, and to shift $t$ so that the critical time comes to the origin. Then $R(t)$ with increasing $\beta$ extends in $t$ in such a way that the results at smaller $|t|$ scale. This is demonstrated in Fig. 3, where we find a reasonable scaling behavior for $N = 16$ with $\kappa = 2.0, 4.0, 8.0$. In fact, supersymmetry of the model plays an important role here [21].

*Summary.*—In this Letter we have studied the nonperturbative dynamics of the Lorentzian matrix model for type IIB superstring theory in ten dimensions. In order to make the model well defined, we introduce the infrared cutoffs on both the spatial and temporal directions. We find that the two cutoffs can be removed in the large-$N$ limit. Moreover, the theory thus obtained has no parameters other than one scale parameter, which is a property expected for nonperturbative superstring theory. The SO(9) symmetry breaks down to SO(3) at some critical time, and the size of the three-dimensional space increases with time. The cosmological singularity is naturally avoided due to noncommutativity.

There are a lot of questions that should be addressed in our model. One of the most urgent questions is whether a local field theory on a commutative space-time appears at the low energy scale. A possible way is to calculate correlation functions of the Wilson loop operators [22]. If this question is answered in the affirmative, we consider it very likely that our model really describes the birth of our Universe from first principles. The next step would be to show that the four fundamental interactions and the matter fields appear in our Universe at a later time.

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[1] A. Vilenkin, Phys. Lett. B **117**, 25 (1982); Phys. Rev. D **30**, 509 (1984); J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[2] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. D **72**, 064014 (2005).
[3] D. Z. Freedman, G. W. Gibbons, and M. Schnabl, AIP Conf. Proc. **743**, 286 (2004); B. Craps, S. Sethi, and E. P. Verlinde, J. High Energy Phys. 10 (2005) 005.
[4] M. Li, Phys. Lett. B **626**, 202 (2005); S. R. Das and J. Michelson, Phys. Rev. D **72**, 086005 (2005); B. Chen, Phys. Lett. B **632**, 393 (2006); J. H. She, J. High Energy Phys. 01 (2006) 002; E. J. Martinec, D. Robbins, and S. Sethi, J. High Energy Phys. 08 (2006) 025; T. Ishino and N. Ohta, Phys. Lett. B **638**, 105 (2006); T. Matsuo, D. Tomino, W. Y. Wen, and S. Zeze, J. High Energy Phys. 11 (2008) 088; D. Klammer and H. Steinacker, Phys. Rev. Lett. **102**, 221301 (2009); J. Lee and H. S. Yang, arXiv:1004.0745.
[5] See R. H. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989) for a scenario based on string-gas cosmology.
[6] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).
[7] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. **B498**, 467 (1997).
[8] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, Nucl. Phys. **B500**, 43 (1997).
[9] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, Prog. Theor. Phys. **99**, 713 (1998).
[10] J. Nishimura and F. Sugino, J. High Energy Phys. 05 (2002) 001.
[11] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo, and S. Shinohara, Nucl. Phys. **B647**, 153 (2002); T. Aoyama and H. Kawai, Prog. Theor. Phys. **116**, 405 (2006).
[12] J. Nishimura, T. Okubo, and F. Sugino, J. High Energy Phys. 10 (2011) 135.
[13] See H. Steinacker, Prog. Theor. Phys. **126**, 613 (2011) for a recent study of new classical solutions in the Lorenzian model.
[14] J. Nishimura and G. Vernizzi, J. High Energy Phys. 04 (2000) 015; Phys. Rev. Lett. **85**, 4664 (2000).
[15] K. N. Anagnostopoulos and J. Nishimura, Phys. Rev. D **66**, 106008 (2002).
[16] W. Krauth, H. Nicolai, and M. Staudacher, Phys. Lett. B **431**, 31 (1998).
[17] P. Austing and J. F. Wheater, J. High Energy Phys. 04 (2001) 019.
[18] T. Yoneya, Prog. Theor. Phys. **97**, 949 (1997).

[19] M. A. Clark, A. D. Kennedy, and Z. Sroczynski, Nucl. Phys. Proc. Suppl. **140**, 835 (2005).

[20] This point of view can be justified in the large-$N$ limit, in which more and more eigenvalues $\alpha$ of $A_0$ appear within a fixed range $|\alpha - t| \leq \delta t$.

[21] S.-W. Kim, J. Nishimura, and A. Tsuchiya (to be published).

[22] M. Fukuma, H. Kawai, Y. Kitazawa, and A. Tsuchiya, Nucl. Phys. **B510**, 158 (1998).