Spin Diffusion in Trapped Clouds of Cold Atoms with Resonant Interactions

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We show that puzzling recent experimental results on spin diffusion in a strongly interacting atomic gas may be understood in terms of the predicted spin diffusion coefficient for a generic strongly interacting system. Three important features play a central role: (a) Fick's law for diffusion must be modified to allow for the trapping potential; (b) the diffusion coefficient is inhomogeneous, due to the density variations in the cloud; and (c) the diffusion approximation fails in the outer parts of the cloud, where the mean free path is long.

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Diffusion in the presence of an external potential is an important problem in diverse fields, ranging from astrophysics, to condensed matter physics, to biology. New vistas for understanding diffusion of spin have been opened up by experiments using resonantly interacting atomic gases [1,2]. These experiments are the analog for spin phenomena of earlier groundbreaking experiments that established that atomic gases may form a perfect fluid with a shear viscosity having the least possible value consistent with quantum mechanics [3,4].

In the spin transport experiments, a cloud of atoms consisting of two hyperfine states of the same atom, which we refer to as $\uparrow$ and $\downarrow$, was studied. Atoms in one state were displaced with respect to those in the other state and the subsequent dynamics was investigated [1,2]. When the population of one hyperfine species is much larger than the other, the diffusive motion is well described by collisional relaxation [2,5]. For equal populations of the $\uparrow$ and $\downarrow$ atoms, previous studies have focussed on the initial bouncing motion of the clouds [6,7]. Here, we analyze the long time scale dynamics and show that, because of the trap potential $V(\mathbf{r})$, Fick's law must be modified. We demonstrate that this effect, combined with the fact that the spin diffusion coefficient is inhomogeneous, leads to predictions for the decay rate that are more than 1 order of magnitude larger than the experimentally measured one in the classical regime. The resolution of this puzzle is shown to be the failure of the diffusion approximation in the outer regions of the cloud. Our analysis accounts for the experimental results in Ref. [1] using the spin diffusion coefficient predicted for a resonantly interacting system. There is a rich variety of regimes for spin relaxation, depending on the trap anisotropy and the density of atoms.

Basic formalism.—In a trap, the magnetization density $M(\mathbf{r}) = n_\uparrow(\mathbf{r}) - n_\downarrow(\mathbf{r})$, where $n_i(\mathbf{r})$ is the density of species i, is not constant in equilibrium. For instance, $M(\mathbf{r}) \propto e^{-V(\mathbf{r})/T}$ for high temperatures T. (We use units in which $k_B = 1$.) Rather the quantity that is constant is the chemical potential difference $\mu_\uparrow - \mu_\downarrow \simeq 2M/\chi$, where $\chi = 2\partial M/\partial(\mu_\uparrow - \mu_\downarrow)$ is the spin susceptibility. Thus diffusion is driven by spatial variations of the chemical potentials, and phenomenologically, the spin current density $\mathbf{j}_M$ is therefore given by the modified Fick's law

$$\mathbf{j}_M = -D\chi\nabla\left(\frac{M}{\chi}\right), \tag{1}$$

where D is the spin diffusion coefficient. Equation (1) reduces to the usual expression $\mathbf{j}_M = -D\nabla M$ when V is constant. We concentrate on the case of temperatures high enough that the gas may be treated using the Maxwell-Boltzmann distribution. In atomic gases, the dominant relaxation process is two-body scattering and, consequently, the diffusion coefficient, which is proportional to the mean free path of a particle, therefore varies inversely with the density $n \propto e^{-V/T}$ [1,8] and $\chi = n/T$, where $n = n_\uparrow + n_\downarrow$.

To determine the diffusive modes, we write $M(\mathbf{r}, t) = e^{-\Gamma t}M(\mathbf{r})$. Insertion of Eq. (1) in the equation of continuity, $\partial_t M + \nabla\cdot\mathbf{j}_M = 0$ gives

$$D_0\nabla^2 P + \Gamma e^{-V/T}P = 0, \tag{2}$$

where D_0 is the diffusion coefficient at the center of the trap ($V = 0$), and $P(\mathbf{r}) = M(\mathbf{r})/n(\mathbf{r})$ is the local fractional polarization. Equation (2) describes diffusion in the presence of an external potential and is often referred to as the Smoluchowski equation [9]. In regions where $V \gg T$, P satisfies the Laplace equation, and therefore the component of P proportional to the spherical harmonic $Y_{\rm lm}$ must vary as r^{-l} in three dimensions since the solution varying as r^l is forbidden by the condition that, by definition, $|P| \le 1$. Thus both P and ∇P vanish as $r \to \infty$. Equation (2) is therefore analogous to the Schrödinger equation for a potential $\propto e^{-V/T}$ and determining the eigenvalue Γ is equivalent to finding the strength of the potential that will produce a zero-energy bound state. Equation (2) may be derived from the variational principle $\delta\Gamma_{\rm var} = 0$, where

$$\Gamma_{\rm var} = \frac{D_0 \int d^3 r (\nabla P)^2}{\int d^3 r e^{-V/T} P^2}. \quad (3)$$

For the lowest mode with a particular symmetry, $\Gamma_{\rm var}$ provides an upper bound on the lowest eigenvalue.

Simple examples.—We first solve (2) for a one-dimensional (1D) harmonic potential, $V = m\omega_z^2 z^2/2$, where ω_z is the trap frequency. For $|z| \to \infty$, P varies as $A + Bz$, where A and B are constants, but because $|P| < 1$, $B = 0$. A numerical solution of (2) for the lowest mode that is odd in z for the boundary condition $P(z) \to$ const for $z \to \infty$ yields for the damping rate the result

$$\Gamma_{\rm 1D} \approx 2.684 \frac{D_0}{l_z^2}, \quad (4)$$

where $l_i^2 = 2k_B T/m\omega_i^2$. Plots of the polarization and the associated spin current density are given in Fig. 1(a). Since (2) is linear, the normalization of P and $\mathbf{j}_M$ in Figs. 1 and 2 is arbitrary. The variational function $\tanh(z/0.7842 l_z)$ gives for $\Gamma_{\rm var}$ the value $2.687 D_0/l_z^2$, which is within $\sim$0.1% of the exact result.

We now consider the spherically symmetric case $V = m\omega^2 r^2/2$. The simplest solution rotationally invariant about the z axis and odd in z has the form $P = Y_{10}(\theta) u(r)/r$ with $Y_{10}(\theta) \propto \cos\theta$. In Fig. 1(b), we plot a numerical solution to (2). Requiring the solution to vanish as $r \to \infty$ yields the damping rate

$$\Gamma = 12.10 \frac{D_0}{l^2}. \quad (5)$$

Figure 2(a) shows contour plots of P and the spin current density, which resembles that for a dipole. The variational function $z/[1 + (r/d)^3]$ has the correct asymptotic behavior for both $r \to 0$ and $r \to \infty$, and it yields $\Gamma_{\rm var} = 12.12 D_0/l^2$ for $d = 0.886 l$.

Anisotropic traps.—The spin diffusion experiments [1] are performed in a prolate trap of the form $V(\mathbf{r}) = m(\omega_\perp^2 \rho^2 + \omega_z^2 z^2)/2 = V_\perp + V_z$, where $\boldsymbol{\rho} = (x, y)$, with

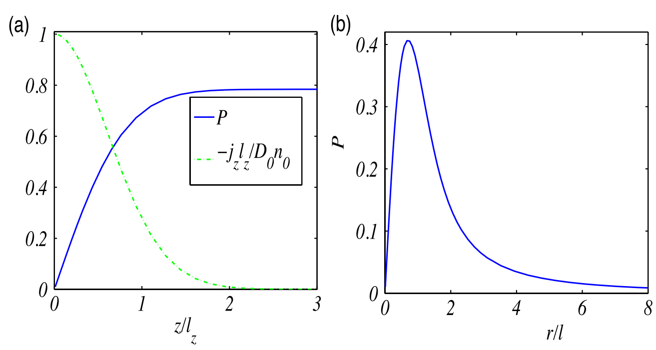

FIG. 1 (color online). (a) The polarization (solid line) and the spin current density (dashed line) for the 1D case. (b) $P(r)$ for the spherical case with $P(\mathbf{r}) = P(r)\cos\theta$.

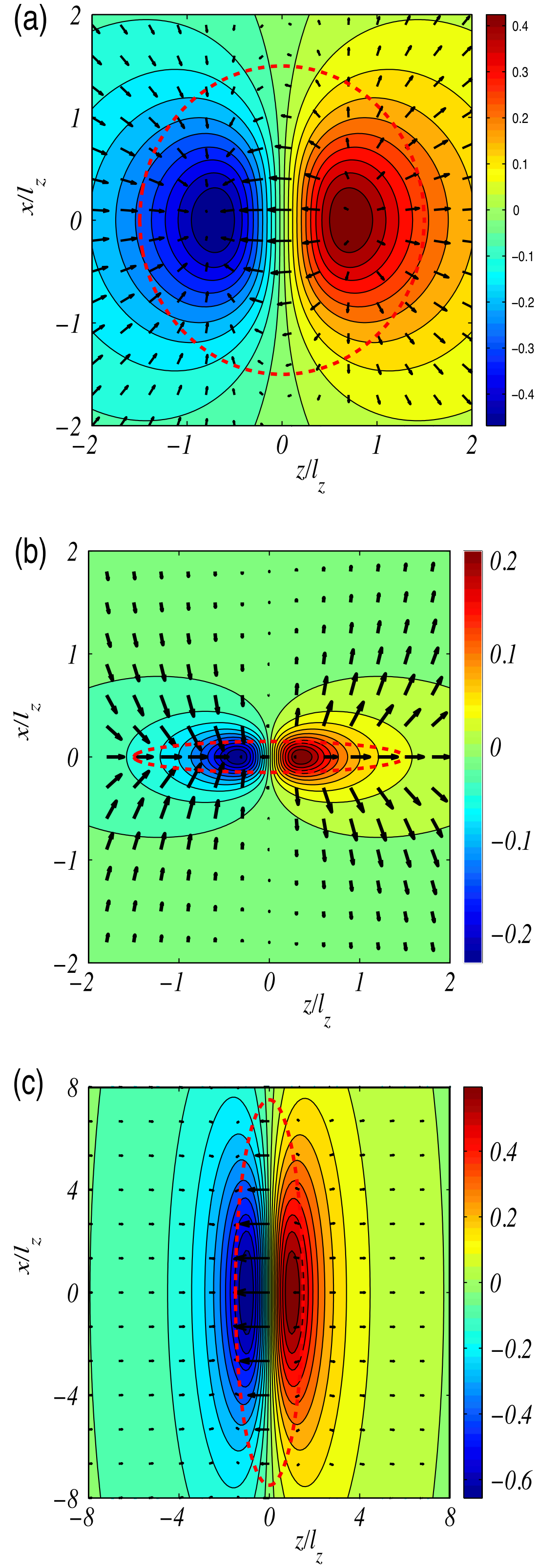

FIG. 2 (color online). Contour plots of the polarization and spin current density (arrows) in the ρz- plane. The red dashed contour shows where the density has fallen to 0.1 of the central value. (a) The spherical case. (b) The prolate case with $\lambda = \omega_z/\omega_\perp = 1/5$. (c) The oblate case with $\lambda = 5$.

$\omega_\perp > \omega_z$. We now solve the diffusion equation (2) for a general aspect ratio $\lambda = \omega_z/\omega_\perp$ using the variational function

$$P(\rho, z) = z/(1 + \tilde{R}^3), \tag{6}$$

with $\tilde{R}^2 = \rho^2/d_\perp^2 + z^2/d_z^2$ which obeys the correct boundary conditions for $r \to 0$ and $r \to \infty$. The variational parameters $d_\perp$ and d_z determine the falloff of the polarization in the transverse and axial directions in units of $l_\perp$ and l_z, respectively. The resulting damping rate is plotted versus λ in Fig. 3. We see that the variational function reproduces very accurately the result for the spherical case $\lambda = 1$. The damping is a decreasing function of λ, since the transverse confinement imposes a gradient in the polarization as is illustrated in Fig. 2. The inset demonstrates that for prolate traps, the length scale of the transverse variations $d_\perp$ becomes longer than $l_\perp$ while the scale of axial variations d_z becomes shorter than l_z; the opposite holds for oblate traps. Figure 2(b) illustrates this important point further for the case $\lambda = 1/5$: the polarization distribution is considerably less prolate than the density distribution, and the current density is significant even in regions where the density is low. Note that, for the prolate and spherical cases, the current has large transverse as well as axial components.

For $\lambda \to \infty$, we see from Fig. 3 that the damping rate approaches the 1D result (4). This reflects the fact that the spin motion becomes 1D with the current essentially in the axial direction from the maximum to the minimum of the polarization, as is clearly seen in Fig. 2(c).

Born-Oppenheimer approximation.—Using the dimensionless variables $\tilde{\boldsymbol{\rho}} = \boldsymbol{\rho}/l_\perp$ and $\tilde{z} = z/l_z$, we see that the

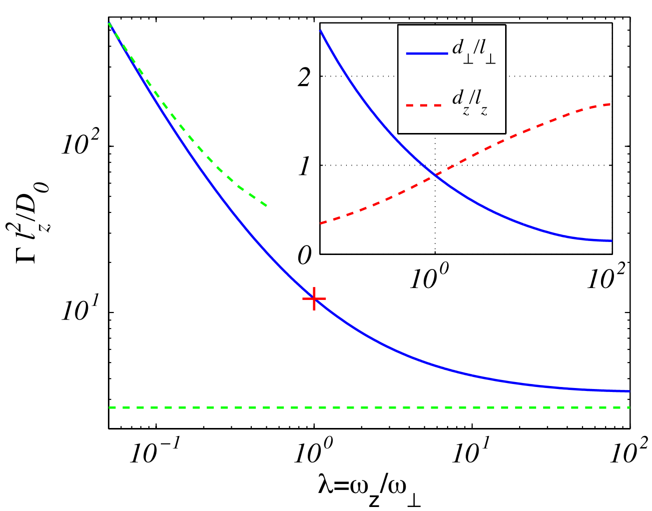

FIG. 3 (color online). The damping rate as a function of the aspect ratio obtained with the variational function (6) (solid line). The result (5) for a spherical trap is plotted as a cross, and the limits $\lambda \to \infty$ (4) and $\lambda \to 0$ (11) as dashed lines. The variational length scales $d_\perp$ and d_z are shown in the inset.

diffusion equation (2) for an anisotropic, harmonic trapping potential is equivalent to a threshold problem in quantum mechanics with an isotropic 3D Gaussian potential, but where the mass for motion in the transverse directions is a factor $\lambda^2 = \omega_z^2/\omega_\perp^2$ smaller than that for axial motion. For the case of a very prolate trap with $\lambda \ll 1$, we can therefore solve the diffusion equation using the Born-Oppenheimer approximation. Writing $P(\mathbf{r}) = \psi(z)\phi(\boldsymbol{\rho}, z)$, we first find the lowest eigenstate for the "light" particle by solving

$$-[\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 - A(z)e^{-\tilde{\rho}^2}]\phi(\rho, z) = E(z)\phi(\rho, z) \tag{7}$$

with $A(z) = A_0 \exp(-\tilde{z}^2)$ and $A_0 = \Gamma l_\perp^2/D_0$; $\phi(z)$ is determined by solving the equation

$$\left[-\frac{\omega_z^2}{\omega_\perp^2}\partial_{\tilde{z}}^2 + E(z)\right]\phi(z) = 0. \tag{8}$$

The damping Γ is determined from the value of A_0 required for Eq. (8) to have a zero-energy bound state that is odd under reflection, $\phi(-z) = -\phi(z)$. To solve (7), we observe that for $\omega_\perp \gg \omega_z$ we expect $\Gamma \ll D_0 l_\perp^{-2}$, which corresponds to $A_0 \ll 1$. The transverse problem then reduces to finding the energy of the lowest bound state in a shallow 2D Gaussian potential $V(\tilde{\rho}) = -A(z)\exp(-\tilde{\rho}^2)$. Such a state always exists and for $V_0 \ll 1$ its energy is $E = -\kappa \exp(4\pi/V_0)$ with $V_0 = \int d^2\tilde{\rho} V(\rho)$ [10,11]. The prefactor is $\kappa = 2\exp(-2\gamma + D_1)$ to lowest order in $A(z)$ where $\gamma \simeq 0.5772$ is Euler's constant and [12]

$$D_1 = \int d^2\tilde{\boldsymbol{\rho}}_1 d^2\tilde{\boldsymbol{\rho}}_2 \frac{V(\tilde{\rho}_1)V(\tilde{\rho}_2)}{V_0^2}\ln(\tilde{\boldsymbol{\rho}}_{12}^2/2) \tag{9}$$

with $\tilde{\boldsymbol{\rho}}_{12} = \tilde{\boldsymbol{\rho}}_1 - \tilde{\boldsymbol{\rho}}_2$. The integrations are straightforward to perform, and we find

$$E(z) = -2e^{-3\gamma}e^{-4/A(z)}. \tag{10}$$

For $A_0 \ll 1$, we can expand the eigenvalue as $E(z) \simeq -2\exp[-3\gamma - 4(1 + \tilde{z}^2)/A_0]$ and when this is inserted in (8), we recover the 1D diffusion problem in a Gaussian trap. Using our 1D result (4), the threshold condition for a bound state odd in z in a very prolate trap becomes

$$\Gamma e^{-4D_0/\Gamma l_\perp^2} = 2e^{3\gamma}\Gamma_{1D}, \tag{11}$$

which is an implicit equation for the damping rate. For $\Gamma \ll 1$, the solution is

$$\Gamma \simeq \frac{4}{\ln(l_z^2/7.6 l_\perp^2)}\frac{D_0}{l_\perp^2}. \tag{12}$$

Thus, the assumption $A_0 \ll 1$ for $l_\perp \ll l_z$ is consistent. However, the numerical factors indicate that the asymptotic expression (12) is a good approximation only for extremely prolate traps. Figure 3 shows that the variational function (6) accurately recovers the prolate limit of the damping rate given by (11).

Failure of the diffusion approximation.—For $\lambda \approx 0.1$ the calculations above predict a damping rate of approximately $200 D_0/l_z^2$, while experimentally the results of Ref. [1] with the expression for D_0 from Ref. [8] give $\Gamma \approx 10 D_0/l_z^2$. We now demonstrate that the discrepancy is due to the failure of the diffusion approximation in the outer parts of the cloud, where the density is low and conditions are collisionless. An approximate expression for the distance r_0 from the z axis at which the diffusion approximation fails may be obtained by arguing that this occurs when a particle has a probability of $1/e^{\alpha}$ of not suffering a collision when it comes in from infinity. Assuming $r_0 \gg l_\perp$, this gives [13]

$$r_0^2 \simeq l_\perp^2 \ln\left(\frac{n_0 \bar{\sigma}}{2\alpha}\sqrt{\frac{k_B T}{2m\omega_\perp^2}}\right). \tag{13}$$

Equation (13) is correct to logarithmic accuracy. The exact value of the parameter $\alpha \sim \mathcal{O}(1)$ may be determined by solving the kinetic equation in the vicinity of the boundary. However, provided $r_0 \gg l_\perp$ the uncertainty in α has little effect on the value of r_0.

At $\rho = r_0$ it is necessary to impose a boundary condition. The flux of atoms in the ρ direction for $\rho > r_0$ is small since, in the absence of collisions, atoms moving to larger values of ρ will be reflected by the trapping potential thereby strongly reducing the net flux. Consequently, we impose the condition that the current in the ρ direction vanish at $\rho = r_0$, or $\partial P/\partial\rho|_{\rho=r_0} = 0$. When the distance to the boundary r_0 is not much larger than the typical length scale l for the spin diffusion modes, it influences the damping rate. In a prolate trap for which $r_0 \gtrsim l_\perp$, the polarization reaches its asymptotic value for $z \to \infty$ for $|z| \lesssim l_z$ and consequently the failure of the diffusion approximation for the motion in the axial direction, which will occur only for large distances compared with l_z, does not affect the results. With the boundary condition $\partial P/\partial\rho|_{\rho=r_0} = 0$, the variational principle derived earlier applies except that the region of integration is limited to $\rho \leq r_0$. For r_0 not too much larger than $l_\perp$, we argue that a good approximation for a trial function is simply a function of z, which gives a damping rate

$$\Gamma \simeq \frac{r_0^2}{l_\perp^2}\Gamma_{1\mathrm{D}}. \tag{14}$$

With increasing r_0, there will be a crossover between the 1D spin currents with a damping given by (14) and the fully 3D hydrodynamic spin currents which have both transverse and axial components as shown in Figs. 1(a) and 1(b) with a damping scaling as $\Gamma \sim D_0/l_\perp^2$. We expect the crossover between the two hydrodynamic solutions to occur when $r_0^2 l_\perp^{-2}\Gamma_{1\mathrm{D}} \sim D_0/l_\perp^2$ which gives $r_0 \sim l_z$.

Comparison with experiment.—We finally compare our results with the experiments in Ref. [1]. For long times, the spin dynamics is determined by the lowest diffusive mode and the observed decay time is related to the damping rate by the relation $\tau = 1/\Gamma$. Using typical experimental numbers reported in Ref. [1], we obtain $3 \lesssim r_0^2/l_\perp^2 \lesssim 6$. This means that the spin dynamics is diffusive in a major part of the cloud, and since $l_z/l_\perp \simeq 10 > r_0/l_\perp$ we expect the motion to be mainly along the z direction with a damping rate given by (14). In Ref. [1], the decay time is written in terms of a spin drag coefficient as $\tau = \Gamma_{\mathrm{sd}}/\omega_z^2$ and we find

$$\frac{\hbar\Gamma_{\mathrm{sd}}}{E_F} = 2\frac{l_\perp^2}{r_0^2}\frac{D_0}{\Gamma_{1\mathrm{D}} l_z^2}\frac{\hbar}{m D_0}\frac{T}{T_F} \simeq \frac{l_\perp^2}{r_0^2} 0.7\sqrt{\frac{T_F}{T}} \tag{15}$$

where we have used (14) and $D_0 \simeq 1.1(T/T_F)^{3/2}\hbar/m$ for a strongly interacting Fermi gas in the classical regime [1,8]. This agrees with the measured high temperature experimental result, $\Gamma_{\mathrm{sd}} = 0.16 E_F \hbar^{-1}\sqrt{T_F/T}$ when $r_{0\perp}^2/l_\perp^2 \simeq 4.4$ which is consistent with the estimate above. In addition to reproducing the magnitude and temperature dependence of the damping, our result also explains the observation that Γ_{sd} is independent of the axial trapping frequency ω_z.

In summary, we have shown that a quantitative account of the measured damping rates of diffusive modes can be given if three novel features of spin diffusion in a trap are taken into account: the failure of Fick's law, the inhomogeneity of the diffusion coefficient, and the failure of the diffusion approximation in the outermost regions of the cloud. The work may be extended in a number of directions: to gases with unequal numbers of the two species, and to degenerate gases, including ones with a condensate of paired fermions. Calculations of damping rates may also be refined by improving variational functions and by obtaining a more quantitative understanding of the boundary between diffusive and collisionless behavior on the basis of the Boltzmann equation.

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