Incoherent Soliton Turbulence in Nonlocal Nonlinear Media

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The long-term behavior of a modulationally unstable nonintegrable system is known to be characterized by the soliton turbulence self-organization process: It is thermodynamically advantageous for the system to generate a large-scale coherent soliton in order to reach the (''most disordered'') equilibrium state. We show that this universal process of self-organization breaks down in the presence of a highly nonlocal nonlinear response. A wave turbulence approach based on a Vlasov-like kinetic equation reveals the existence of an incoherent soliton turbulence process: It is advantageous for the system to self-organize into a large-scale, spatially localized, incoherent soliton structure.

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Introduction.—Understanding the mechanisms responsible for self-organization processes in conservative and reversible Hamiltonian wave systems is an arduous problem that generated a significant interest. An important achievement was accomplished when Zakharov and collaborators identified a soliton turbulence self-organization process in the framework of the focusing nonintegrable nonlinear Schrödinger (NLS) equation [1]. Starting from a homogeneous initial state, the wave exhibits a modulational instability (MI) process that leads to the formation of a train of solitonlike pulses. Because of the nonintegrable nature of the interaction, the solitons interact inelastically, so that the system irreversibly relaxes toward a statistical equilibrium state, in which a large-scale coherent soliton remains immersed in a sea of thermalized small-scale fluctuations [2–4]. The coherent soliton then plays the role of a ''statistical attractor'' for the Hamiltonian system, while the small-scale fluctuations contain, in principle, all information necessary for time reversal. It is important to underline that the soliton solution realizes the minimum of the energy (Hamiltonian): The system then relaxes toward the state of lowest energy, which allows the small-scale fluctuations to store the maximum amount of kinetic energy $\mathcal{E}$ [2–4]. In this regard, this self-organization process has, in essence, a thermodynamic origin: It is thermodynamically advantageous for the system to generate a coherent soliton structure, because this allows the system to increase the amount of disorder in the small-scale fluctuations [2–4].

The idea that an increase of entropy in a nonintegrable Hamiltonian system requires the generation of a large-scale coherent structure is fundamental. For instance, in the presence of a defocusing NLS interaction, the self-organization process manifests itself by means of the generation of a coherent plane wave in the midst of thermalized fluctuations [5,6]. This phenomenon of condensation of classical nonlinear waves [6] generated much interest [7,8], in relation with Bose-Einstein condensation in dilute quantum gases [9].

Our aim in this Letter is to show that this universal process of self-organization breaks down in the presence of a highly nonlocal nonlinear response. A nonlocal wave interaction means that the response of the nonlinearity at a particular point is not determined solely by the wave intensity at that point, but also depends on the wave intensity in its vicinity. Nonlocality thus constitutes a generic property of a large number of nonlinear wave systems [10–14]. In contrast with the expected soliton turbulence scenario, we show here that a highly nonlocal wave system self-organizes into spatially localized incoherent structures, i.e., ''incoherent solitons'' (ISs). These IS structures are of a fundamentally different nature than the ''incoherent optical solitons,'' which were recently investigated in various different circumstances [15–22]. A wave turbulence (WT) [4,5,23,24] approach of the problem reveals that this type of IS can be described in detail in the framework of a Vlasov-like kinetic equation, which is shown to provide an ''exact'' statistical description of the highly nonlocal random wave system. In particular, we obtain an IS solution characterized by a compactly supported spectrum, which is found in quantitative agreement with the NLS simulations. This kinetic formulation reveals the existence of an ''incoherent soliton turbulence'' process, in which ISs irreversibly coalesce into a single large IS: an increase of ''disorder'' in the system requires the generation of an IS structure. Furthermore, contrarily to the conventional WT approach, the Vlasov equation derived here does not require the assumption of weakly nonlinear interaction, a feature that may shed new light on the important issue of strong turbulence.

Model.—A nonlocal nonlinear response is found in several systems such as, e.g., dipolar Bose-Einstein condensates [10], atomic vapors [11], nematic liquid crystals [12], thermal susceptibilities [13], and plasma physics [14]. We consider here the standard nonlocal NLS model equation,

$$i\partial_t\psi + \beta\partial_{xx}\psi + \gamma\psi\int_{-\infty}^{+\infty} U(x - x')|\psi|^2(t, x')dx' = 0, \tag{1}$$

where $U(x)$ is the nonlocal response, while β and γ refer to the linear (dispersive) and nonlinear coefficients, respectively. Equation (1) conserves the power (or number of particles) $\mathcal{N} = \int|\psi|^2dx$, and the Hamiltonian $\mathcal{H} = \mathcal{E} + \mathcal{U}$, where $\mathcal{E} = \beta\int|\partial_x\psi|^2dx$ and $\mathcal{U} = -\frac{\gamma}{2}\iint U(x - x')|\psi(x)|^2|\psi(x')|^2dxdx'$ denote the kinetic and nonlinear contributions to the energy $\mathcal{H}$. We denote by σ the spatial extension of $U(x)$. The dynamics will be shown to be ruled by the comparison of σ with the healing length $\Lambda = \sqrt{\beta/(\gamma\rho)}$, where $\rho = \mathcal{N}/L$ is the density of power (particles), L being the size of the numerical window. We underline that, although we consider here a 1D model, our work can readily be extended to any spatial dimension.

NLS simulations.—A physical insight into incoherent soliton turbulence may be obtained by integrating numerically the NLS Eq. (1). The initial condition is a homogeneous (plane) wave $\psi(x, t = 0) = \sqrt{\rho}$ with a superimposed small noise to initiate the MI process ($\beta\gamma > 0$). In this example, we considered a Gaussian response, $U(x) = \exp[-x^2/(2\sigma^2)]/\sqrt{2\pi\sigma^2}$; however, the same behavior is obtained, e.g., with an exponential response. The remarkable result is that the behavior of the system changes in a drastic way as the ratio σ/Λ increases. For $\sigma < \Lambda$, we recover the conventional soliton turbulence scenario, in which the solitons generated by MI interact inelastically and slowly merge into a big coherent soliton that remains immersed in a sea of small-scale fluctuations [see Fig. 1(a)]. These fluctuations exhibit a slow thermalization process, characterized by an irreversible evolution toward an equilibrium state of energy equipartition [2,3], as described by the Hasselmann WT kinetic equation and an analogy of Boltzmann's H theorem of entropy growth [4,5,8,23]. Conversely, for $\sigma \gg \Lambda$, the system no longer generates a coherent soliton but instead self-organizes into an IS-like structure, whose typical width, Δ, is of the same order as the nonlocal range, $\sigma \sim \Delta$ [see Figs. 1(b) and 2]. More precisely, the ISs generated through MI coalesce into a unique IS, so that the final IS captures almost all the power of the wave, $\mathcal{N}_{\mathrm{IS}} \sim \mathcal{N}$. Since $\sigma \sim \Delta$, the amplitude of the IS increases proportionally with the grid size L. We underline that the IS is characterized by a compactly supported spectrum [see Fig. 3(b)], in marked contrast with the expected thermal equilibrium spectrum, whose tails exhibit an energy equipartition power law, $|\tilde{\psi}|^2(k) \sim k^{-2}$ [2].

To qualitatively interpret this result, we recall that a coherent soliton results from a balance between the linear and the nonlinear effects, so that its width l and amplitude a are related through $al \sim \sqrt{\beta/\gamma}$. In order for the soliton to be confined by its self-consistent potential, the width l should be of the order of, or larger than, σ (otherwise, the self-consistent potential would be smoothed out by the large nonlocality). The amplitude must thus satisfy $a < \sqrt{\beta/\gamma}/\sigma$, i.e., $a/\sqrt{\rho} < \Lambda/\sigma$. In the regime of incoherent soliton turbulence, we have $\sigma \gg \Lambda$, so that $a \ll \sqrt{\rho}$: If the soliton were generated, its amplitude would be smaller than the amplitude of the fluctuations ($\sim \sqrt{\rho}$); i.e., the small-amplitude soliton would not be able to feel its self-induced potential, which explains why the coherent soliton is not generated.

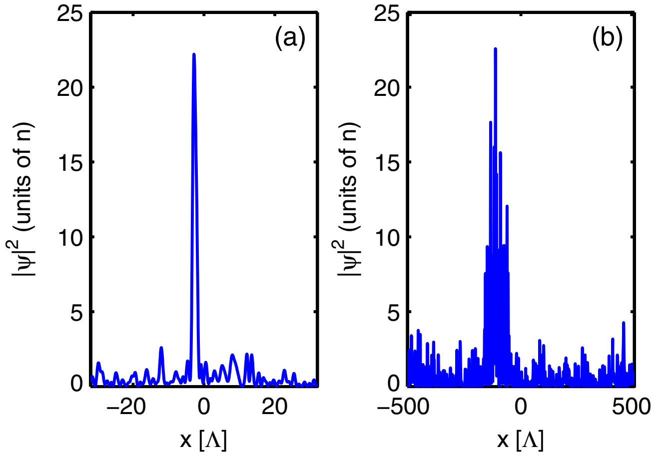

FIG. 1 (color online). Numerical simulations of the NLS Eq. (1) starting from a homogeneous wave amplitude, $\psi(x, t = 0) = \sqrt{n}$. For (a) $\sigma = \Lambda$, one recovers the conventional soliton turbulence scenario ($t = 600\tau_0$). For (b) $\sigma = 30\Lambda$, the wave self-organizes into an IS state ($t = 7000\tau_0$). [$\tau_0 = 1/(\gamma\rho)$, $\Lambda = \sqrt{\beta/(\gamma\rho)}$, and $\rho = \mathcal{N}/L$; we applied periodic boundary conditions].

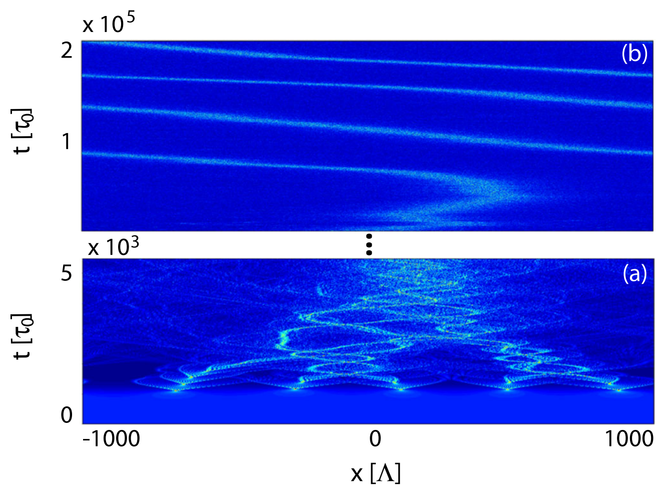

FIG. 2 (color online). Spatiotemporal evolution of the wave intensity $|\psi|^2(x, t)$ obtained by integrating numerically the NLS Eq. (1) for $\sigma = 10^2\Lambda$: The ISs generated through coherent MI coalesce into a large-scale IS structure that is conserved for arbitrarily large times.

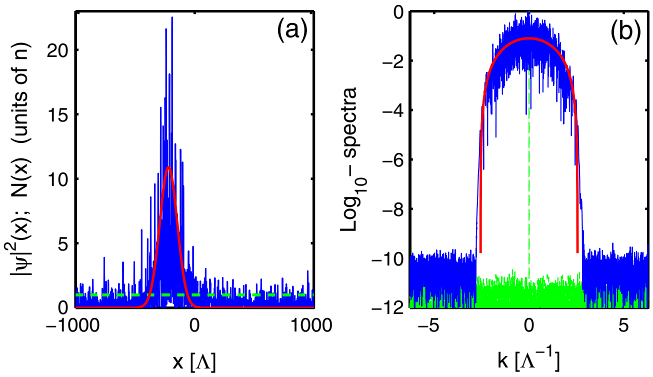

FIG. 3 (color online). Comparison of the simulations of the NLS Eq. (1) with the stationary analytical solution (4) of the Vlasov Eq. (3): (a) Intensity $|\psi|^2(x, t_0)$ at $t_0 = 160\,000\tau_0$ corresponding to Fig. 2 (thin blue line), and intensity profile $N^{\rm st}(x) = (2\pi)^{-1}\int n_k^{\rm st}(x)dk$ associated to the solution (4) with $\sigma_N = 75\Lambda$ (thick red line). (b) Spectrum $|\tilde{\psi}|^2(k, t_0)$ (thin blue line) corresponding to (a), and Vlasov spectrum $S^{\rm st}(k) = \int n_k^{\rm st}(x)dx$ associated to the solution (4) in the Log_{10} scale (thick red line). The dashed green lines show the initial conditions: (a) $\psi(x, t = 0)$, (b) $|\tilde{\psi}|^2(k, t = 0)$. The compactly supported form of the spectrum in (b) is analogous to that obtained in Fig. 4(h).

Despite the fact that a coherent soliton cannot be generated, it turns out that the formation of a localized IS structure leads to an increase of "disorder" in the system, i.e., an increase of the kinetic energy $\mathcal{E}$, which provides a measure of the "amount of fluctuations" in the system [1–4]. We compare the kinetic and nonlinear contributions to the total energy in different states of the system. In the initial coherent state (a), $\psi^{(a)}(x, t = 0) = \sqrt{\rho}$, we have $\mathcal{E}^{(a)} = 0$, so that $\mathcal{H}^{(a)}/L = -\gamma\rho^2/2$. We first note that this state (a) cannot evolve toward a statistical homogeneous (i.e., nonlocalized) incoherent state (b). Indeed, because of the averaging of the fluctuations due to the nonlocal response, we would have $\mathcal{H}^{(b)}/L = \rho\beta/\lambda_c^2 - \gamma\rho^2/2 > \mathcal{H}^{(a)}/L$, where λ_c denotes the correlation length of the wave. Accordingly, the evolution $(a) \to (b)$ cannot preserve the energy and is thus forbidden. Conversely, the formation of an IS is possible and "thermodynamically favorable": Since the IS captures almost all the power of the wave ($\mathcal{N}_{\rm IS} \sim \mathcal{N}$) and since $\Delta \sim \sigma$, the intensity profile of the IS can be written as $N(x) = \langle|\psi|^2\rangle \times (x) \sim \rho(L/\sigma)N_0(x/\sigma)$ and $U(x) = \sigma^{-1}U_0(x/\sigma)$, with $\int U_0(\tilde{x})d\tilde{x} = \int N_0(\tilde{x})d\tilde{x} = 1$ ($\tilde{x} = x/\sigma$). This transformation allows us to extract a dimensionless number, $C_0 = \int N_0(\tilde{x})U_0(\tilde{x} - \tilde{x}')N_0(\tilde{x}')d\tilde{x}d\tilde{x}'/2$, of order 1 in the expression of the Hamiltonian, $\mathcal{H}^{(c)}/L = \rho\beta/\lambda_c^2 - \gamma\rho^2C_0L/\sigma$. Because $L \gg \sigma$, this shows that the formation of an IS leads to a decrease of the contribution of the nonlinear energy. This decay is compensated by an increase in the kinetic energy: the evolution $(a) \to (c)$ leads to an increase of $\Delta\mathcal{E}/L \sim \rho\beta/\lambda_c^2 = -\Delta\mathcal{U}/L \sim \gamma\rho^2C_0L/\sigma$. Accordingly, the generated IS is characterized by $\mathcal{E}^{(c)} \sim \mathcal{U}^{(c)}$ and by a typical correlation length

$$\lambda_c \sim \sqrt{\beta\sigma/(\gamma\rho LC_0)} \sim \Lambda\sqrt{\sigma/L}. \qquad (2)$$

The IS structures identified numerically in Figs. 1 and 2 are fundamentally different than the spatial IS widely investigated in optics [15–19]. Indeed, we deal here with an instantaneous response of the nonlinearity [see Eq. (1)], whereas the spatial self-trapping of an incoherent beam is only possible because of the noninstantaneous nonlinearity, which averages the wave fluctuations provided that its response time τ_R is much longer than the correlation time t_c of the wave, i.e., $\tau_R \gg t_c$ [15–19]. The ISs investigated here are also different from those recently demonstrated in "effectively instantaneous" nonlocal nonlinear materials [20].

Vlasov approach.—The simulations of the NLS Eq. (1) in Figs. 1 and 2 correspond to a single realization of the stochastic function $\psi(x, t)$. We adopt a statistical approach to unveil the underlying soliton behavior based on a WT-type theory. The classical WT theory provides a natural closure of the infinite hierarchy of moment equations for the random wave ψ on the basis of the fundamental assumption that the nonlinearity is weak, $|\mathcal{U}/\mathcal{E}| \ll 1$ [4,5,23,24]. In the weakly nonlinear regime, linear dispersive effects dominate the interaction and bring the wave to a state of Gaussian statistics. If the random wave exhibits fluctuations that are homogeneous in space, the WT theory leads to the irreversible Hasselmann equation that describes wave thermalization to equilibrium [4,5,23,25]. Conversely, if the wave exhibits a quasihomogeneous statistics, the WT theory leads to a reversible Vlasov-like equation for the local spectrum of the field [24,25], which is defined from $n_k(x, t) = \int B(x, \xi, t)\exp(-ik\xi)d\xi$, where $B(x, \xi, t) = \langle\psi(x + \xi/2, t)\psi^*(x - \xi/2, t)\rangle$ is the correlation function.

We stress that the validity of the kinetic equation that we derive here does not require the assumptions of (i) weakly nonlinear interaction and of (ii) quasihomogeneous statistics. Indeed, in the highly nonlocal regime defined by the small parameter $\varepsilon = \Lambda/\sigma \ll 1$, the length scale of the inhomogeneous statistics (i.e., width of an IS) is of the same order as the nonlocal range, $\sigma \sim \Delta$, and we have $\lambda_c \ll \sigma$, i.e., the quasihomogeneous statistics is automatically satisfied (note that, according to Eq. (2), we have $\lambda_c/\Delta \sim \varepsilon\sqrt{\sigma/L} \ll 1$). In these conditions, the self-averaging property of the nonlinear response holds, which leads to a closure of the hierarchy of moment equations. More specifically, using statistical arguments similar to those in [26], one can show that, owing to the highly nonlocal response, the statistics of the random wave turns out to be Gaussian. In this sense, the Vlasov equation derived here provides an "exact" statistical description of the random field $\psi(x, t)$ in the highly nonlocal regime,

$\varepsilon \ll 1$. Using a multiscale series expansion in the small parameter ε, we obtain the Vlasov-like equation [27]

$$\partial_t n_k(x,t) + \partial_k \tilde{\omega}_k(x,t)\partial_x n_k(x,t) - \partial_x \tilde{\omega}_k(x,t)\partial_k n_k(x,t) = 0, \quad (3)$$

where the generalized dispersion relation reads $\tilde{\omega}_k(x,t) = \omega(k) - V(x,t)$, with $V(x,t) = \gamma \int U(x-x')N(x')dx'$, $N(x,t) = B(x,\xi=0,t) = (2\pi)^{-1}\int n_k(x,t)dk$, and $\omega(k) = \beta k^2$. Equation (3) conserves $\mathcal{N} = (2\pi)^{-1}\int n_k(x,t)dkdx$, the Hamiltonian $\mathcal{H}_{\mathrm{Vl}} = \int \omega(k)n_k(x,t)dkdx - \frac{1}{2}\int V(x)N(x)dx$, and $\mathcal{M} = \int f[n]dkdx$, where $f[n]$ is an arbitrary functional of n. The kinetic Eq. (3) has a structure analogous to a Vlasov equation considered in plasma physics [28] or more recently to studying long-range interacting systems [29]. However, to our knowledge, it is the first time that this equation is considered in nonlinear wave systems described by NLS-like equations. In particular, it differs from the Vlasov equation used to describe ISs and incoherent MIs in plasmas [24,30], hydrodynamics [31], and optics [17,19,22,26].

IS solution.—Soliton solutions to Vlasov-like equations have not been widely investigated in the literature, presumably because of the strong assumptions (i) and (ii) discussed above, which seriously limit the physical relevance of the Vlasov equation. However, contrary to the conventional Vlasov equation [17,19,24,30,31], here Eq. (3) is expected to provide an "exact" description of the ISs identified numerically in Fig. 2. We generalize to a nonlocal interaction and to a moving solution the standing IS solution obtained in [30] for a local interaction, $U(x) \to \delta(x)$. In a reference frame traveling with a velocity v, the stationary Eq. (3) reads $\partial_k \tilde{\omega}_k(z)\partial_z n_k(z) - \partial_z \tilde{\omega}_k(z)\partial_k n_k(z) + v\partial_z n_k(z) = 0$, with $z = x + vt$. Assuming a Gaussian intensity profile, $N(z) = \mathcal{N}(2\pi\sigma_N^2)^{-1/2}\exp[-z^2/(2\sigma_N^2)]$, and, making use of the self-consistent method [30,32], we obtain the exact IS solution of the Vlasov Eq. (3)

$$n_k^{\mathrm{st}}(x+vt) = Q_\alpha\{c_\alpha N^\alpha(x+vt) - \beta[k + v/(2\beta)]^2\}^{1/\alpha - 1/2}, \quad (4)$$

where $Q_\alpha = 2\pi\beta^{1/2}\Gamma(\alpha^{-1}+1)/[\Gamma(\alpha^{-1}+1/2)\times \Gamma(1/2)c_\alpha^{1/\alpha}]$, $\Gamma(x)$ being the Gamma function, and $c_\alpha = (2\pi)^{\alpha/2-1/2}\gamma\sigma_N^\alpha\mathcal{N}^{1-\alpha}(\sigma^2+\sigma_N^2)^{-1/2}$, with $\alpha = [1 + (\sigma/\sigma_N)^2]^{-1}$. As remarkably illustrated in Fig. 3, the solution (4) is found in quantitative agreement with the numerical simulations of the NLS Eq. (1). This good agreement is obtained for $|\mathcal{U}/\mathcal{E}| \simeq 1.5$, which concords with the qualitative estimation discussed above through Eq. (2) [i.e., $\mathcal{U}^{(c)} \sim \mathcal{E}^{(c)}$] and thus confirms the validity of the Vlasov approach in the nonlinear regime of interaction. σ_N only weakly affects the analytical spectrum $S^{\mathrm{st}}(k) = \int n_k^{\mathrm{st}}(x)dx$ in Fig. 3(b). Note that the typical width k_c of the compactly supported spectrum of the IS solution (4) is given by a self-trapping condition, $\beta k^2 - V(x) \le 0$, i.e., $k_c \sim \sqrt{V/\beta}$, where $V(x) = \gamma U * N = (\gamma\rho L/\sigma)\times \int U_0(\tilde{x}-\tilde{x}')N_0(\tilde{x}')d\tilde{x}' \sim \gamma\rho L/\sigma$. Accordingly, $\lambda_c \sim k_c^{-1} \sim \Lambda\sqrt{\sigma/L}$, which is in agreement with the estimation given in Eq. (2). Also note that the small soliton velocity $v \sim 0.05\Lambda\tau_0^{-1}$ in Fig. 2(b) gives a negligible frequency shift $k_s = -v/(2\beta)$ in the spectrum $S^{\mathrm{st}}(k)$, i.e., $k_s \ll k_c$. We remark that the momentum captured by the IS has mean zero and a standard deviation that grows like $\langle P_{\mathrm{IS}}^2\rangle^{1/2} \sim \sqrt{L}$ as a function of the grid size L (as a consequence of the central limit theorem for fluctuation phenomena), while the power captured by the soliton grows like $\mathcal{N}_{\mathrm{IS}} \sim L$, and therefore the fluctuations of the velocity of the IS should decay as $\langle V_{\mathrm{IS}}^2\rangle^{1/2} = \langle P_{\mathrm{IS}}^2\rangle^{1/2}/\mathcal{N}_{\mathrm{IS}} \sim 1/\sqrt{L}$.

We finally show that the Vlasov Eq. (3) describes a process of incoherent soliton turbulence, in agreement with the NLS simulation reported in Fig. 2(a) ($t > 1000\tau_0$): The IS structures generated through coherent MI collide and slowly merge into a single large IS. This scenario is reproduced in detail by the Vlasov Eq. (3) (see Fig. 4). We started the simulation from a homogeneous spectrum, $n_k(x, t=0) = n_k^0$, which is periodically perturbed to seed the incoherent MI [15–17,19,30,31]. Because of the nonlinear Hamiltonian flow, particles following different orbits travel at different angular speeds in the phase space (x, k), a process known as phase-mixing.

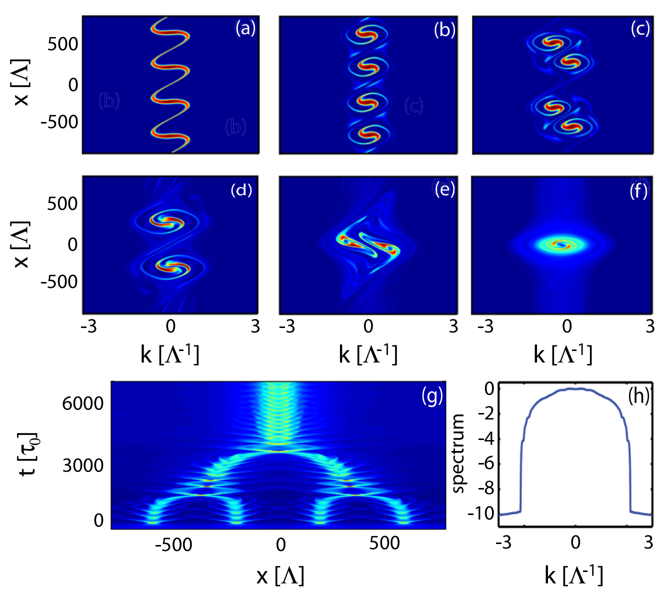

FIG. 4 (color online). Simulation of the Vlasov Eq. (3) for $\sigma = 10^2\Lambda$. The initial homogeneous spectrum exhibits incoherent MI: the four modulations excited by the initial condition lead to the generation of four ISs, which slowly coalesce into two, and then into a single IS structure. (a) $t = 300$, (b) $t = 1000$, (c) $t = 1500$, (d) $t = 3000$, (e) $t = 4000$, and (f) $t = 10^4$ (in units of τ_0). (g) Evolution of the intensity, $N(x,t) = (2\pi)^{-1}\times \int n_k(x,t)dk$. (h) Spectrum $S(k,t_0) = \int n_k(x,t_0)dx$ at $t_0 = 7000\tau_0$ (Log_{10} scale).

Each MI modulation thus starts spiraling in the phase space, then leading to the formation of four ISs, which are mutually attracted and coalesce into two, and eventually into a single large IS (Fig. 4). We remark that the spectrum $S(k)$ in Fig. 4(h) has the same compactly supported form as the analytical soliton solution (4) and the NLS simulations in Fig. 3(b). Note that phase-mixing is responsible for a homogenization of the perturbations of localized Vlasov states, a dynamical feature that refers to a long-standing mathematical problem [33].

Conclusion.—We reported a process of incoherent soliton turbulence in highly nonlocal wave systems that is described in detail by a Vlasov-like kinetic approach. A preliminary study indicates that our results can be extended to account for an additional defocusing local nonlinearity relevant to dipolar Bose-Einstein condensates [10]. Work is also in progress to study genuine long-range interacting nonlinear wave systems [29].

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