Josephson Junction with a Magnetic-Field Tunable Ground State

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We consider an asymmetric 0-π Josephson junction consisting of 0 and π regions of different lengths L_0 and L_π. As predicted earlier this system can be described by an effective sine-Gordon equation for the spatially averaged phase ψ so that the effective current-phase relation of this system includes a *negative* second harmonic $\propto \sin(2\psi)$. If its amplitude is large enough, the ground state of the junction is doubly degenerate $\psi = \pm\varphi$, where φ depends on the amplitudes of the first and second harmonics. We study the behavior of such a junction in an applied magnetic field H and demonstrate that H induces an additional term $\propto H \cos\psi$ in the effective current-phase relation. This results in a nontrivial ground state *tunable* by magnetic field. The dependence of the critical current on H allows for revealing the ground state experimentally.

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Josephson junctions (JJs) with a phase shift of π in the ground state [1] attracted a lot of interest in the recent years [2–7]. These JJs can be used as on-chip phase batteries for biasing various classical [8] and quantum [9] circuits. This allows for removing external bias lines and reducing decoherence. Currently, it is possible to fabricate simultaneously both 0 and π JJs using various technologies such as superconductor-ferromagnet heterostructures [10] or JJs based on d-wave superconductors [11–13].

It would be remarkable to have a JJ (a *phase battery*) providing an *arbitrary* phase shift φ, rather than just 0 or π. Long arrays of 0-π-0-π-... JJs with short segments were suggested as systems, where the φ JJ can be realized [14–16]. Each of the segments are assumed to have a standard current-phase relation (CPR) $j_s = \pm j_0 \sin\phi$. The phase ϕ in these systems can be written as a sum of two terms $\phi(x) = \psi + \xi(x)\sin\psi$, where ψ is a constant spatially averaged phase and $|\xi(x)| \ll 1$ is alternating on a scale of the order of the segment's length. The effective CPR then reads [14–16]

$$j_s = j_1 \sin(\psi) + j_2 \sin(2\psi), \tag{1}$$

where $j_1 = \langle j_c(x) \rangle$ is the spatially averaged critical current density and $j_2 < 0$ is the amplitude of the effective second harmonic. We note that special stochastic distributions of facets may also lead to $j_2 > 0$ [17]. The value of j_2 depends on parameters of the junctions. For $j_2 < -j_1/2$ the ground state of the system is doubly degenerate with $\psi = \pm\varphi$, where

$$\varphi = \arccos[-j_1/(2j_2)]. \tag{2}$$

This φ JJ [16] is the generalization of a π JJ and can provide an *arbitrary* phase bias $0 < \varphi < \pi$. These JJs have unusual physical properties [18], e.g., non quantized

Josephson vortices that were predicted [14] and observed experimentally [15].

The simplest system, which is suitable for the realization of a "short" φ JJ, is an asymmetric (i.e., $L_0 \neq L_\pi$) 0-π JJ with relatively short segments of length $L_0, L_\pi \lesssim \lambda_J$, where $\lambda_J \propto 1/\sqrt{j_0}$ is the Josephson length. Such a system can be considered as one period of an infinitely long 0-π-0-π-... chain. The best method to reveal the φ JJ experimentally is to include it in a superconducting loop (thus forming a SQUID) and to measure the spontaneously generated flux. Another option is to study these JJs in a magnetic field H. It is worth mentioning that there is an apparent contradiction here. On one hand, the usual linear phase ansatz $\phi(x) = Hx + \phi_0$ in asymmetric 0-π JJ results in a critical current $I_c(H)$ with a cusplike *minimum* at $H = 0$, see Eq. (11) in Ref. [11] as well as Refs. [19,20]. This $I_c(H)$ should be mirror symmetric with respect to the I_c axis, i.e., $I_c(-H) = I_c(+H)$. On the other hand, the presence of the second harmonic in the CPR (1) and the linear phase ansatz for ψ leads to a Fraunhofer-like $I_c(H)$ with the *maximum* at $H = 0$ [18]. Moreover, when $|j_2| > 0.5 j_1$ there are two critical currents $I_{c\pm}(H)$, corresponding to depinning of the phase from different potential wells, but both of the $I_{c\pm}(H)$ dependences have the main maximum at $H = 0$ [18].

In this paper we study the asymmetric 0-π JJ in a magnetic field in detail. We demonstrate that the amplitude of the second harmonic j_2 in Eq. (1) depends only on the JJ parameters, while the magnetic field H induces an additional term $\propto H \cos\psi$. Further, we predict the dependence of $I_c(H)$ at small magnetic fields.

Consider an asymmetric JJ shown in Fig. 1. The dependence $j_c(x)$ is given by

$$j_c(x) = -j_0, \quad \text{if} -L_\pi < x < 0; \tag{3a}$$
$$j_c(x) = +j_0, \quad \text{if } 0 < x < L_0 \tag{3b}$$
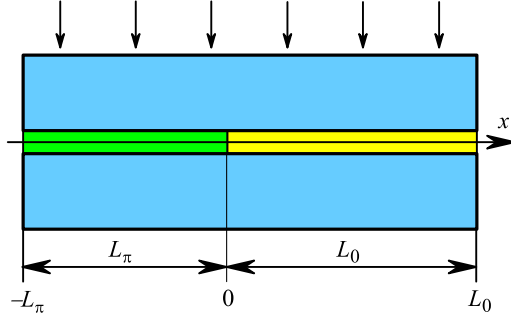
FIG. 1 (color online). Geometry of the 0-π JJ.

The behavior of the phase $\phi(x)$ is defined by the static
sine-Gordon equation

$$\phi''(x) - j_c(x)\sin[\phi(x)] = -\gamma, \qquad (4)$$

where the coordinate x and lengths $L_{0,\pi}$ are normalized in
the usual way to λ_J and $\gamma = j/j_0$ is the normalized bias
current density. $j_c(x)$ below is also normalized to j_0. Next,
we write [14]

$$j_c(x) = \langle j_c\rangle[1 + g(x)], \qquad (5)$$

where

$$\langle j_c\rangle = \frac{1}{L_0 + L_\pi}\int_{-L_\pi}^{+L_0} j_c(x)dx = j_0\frac{L_0 - L_\pi}{L_0 + L_\pi} \qquad (6)$$

is the average value of the critical current density and

$$g(x) = \begin{cases} g_\pi = \frac{-2L_0}{L_0 - L_\pi}, & \text{for } x < 0; \\ g_0 = \frac{+2L_\pi}{L_0 - L_\pi}, & \text{for } x > 0, \end{cases} \qquad (7)$$

is the deviation from the average value ($\langle g(x)\rangle = 0$).
   Thus, we rewrite Eq. (4) in terms of $g(x)$ and arrive to

$$\phi'' - \langle j_c\rangle[1 + g(x)]\sin\phi = -\gamma. \qquad (8)$$

In what follows we treat short JJs ($L_0, L_\pi \lesssim 1$). Therefore,
the solution $\phi(x)$ can be sought in the form [14]

$$\phi(x) = \psi + \xi(x)\sin\psi, \qquad (9)$$

where ψ is a constant [21] and $\xi(x)$ describes small varia-
tions of the phase around ψ, i.e., $|\xi(x)| \ll 1$, $\langle\xi(x)\rangle = 0$.
Substituting the ansatz (9) into Eq. (8) and expanding to the
first order in $\xi(x)$ we get

$$\xi''\sin\psi - \langle j_c\rangle[1 + g(x)][1 + \xi(x)\cos\psi]\sin\psi = -\gamma. \qquad (10)$$

Equation (10) has two types of terms: the constant ones and
varying ones. Note, that the term $g(x)\xi(x)$ has both the
constant (average) part $\langle g(x)\xi(x)\rangle$ and the deviation from
the average $g(x)\xi(x) - \langle g(x)\xi(x)\rangle$. Thus, from Eq. (10) the
relation for the constant terms reads

$$\gamma = \langle j_c\rangle[\sin\psi + \langle g(x)\xi(x)\rangle \sin\psi\cos\psi]. \qquad (11)$$

The equation for $\xi(x)$, recalling definition (5), is

$$\xi'' - j_c(x)\cos\psi\,\xi(x) = \langle j_c\rangle[g(x) - \langle g\xi\rangle\cos\psi], \qquad (12)$$

It turns out that both terms $\propto \cos\psi$ have an extremely
weak influence on the results. Therefore, for the sake of
simplicity, we omit them right away to arrive to

$$\xi'' = \langle j_c\rangle g(x). \qquad (13)$$

Solutions of this equation include four constants to be
determined from the matching conditions at $x = 0$

$$\xi_\pi(0) = \xi_0(0); \qquad \xi'_\pi(0) = \xi'_0(0), \qquad (14)$$

and boundary conditions at $x = -L_\pi$ and $x = L_0$

$$\xi'_\pi(-L_\pi)\sin\psi = h; \qquad \xi'_0(L_0)\sin\psi = h, \qquad (15)$$

where $h = 2H/H_{c1}$ is the normalized applied mag-
netic field, $H_{c1} = \Phi_0/(\pi\lambda_J\Lambda)$ is the penetration field
and $\Lambda \approx 2\lambda_L$ is the effective magnetic thickness of the
JJ. Thus, we arrive to the expression for $\xi(x)$ and can
calculate

$$\langle g\xi\rangle = \Gamma_0 + \Gamma_h\frac{h}{\sin\psi}, \qquad (16)$$

where the coefficients

$$\Gamma_0 = -\frac{4}{3}\frac{L_0^2 L_\pi^2}{L_0^2 - L_\pi^2}; \qquad \Gamma_h = \frac{L_0 L_\pi}{L_0 - L_\pi}. \qquad (17)$$

   Thus, Eq. (11) gives the effective CPR of the JJ

$$\gamma = \langle j_c\rangle\left[\sin\psi + \Gamma_h h\cos\psi + \frac{\Gamma_0}{2}\sin(2\psi)\right]. \qquad (18)$$

The result is remarkable—as in earlier works [14–16] one
gets the second harmonic $\sin(2\psi)$ with the negative am-
plitude of $\Gamma_0/2$. In addition, the magnetic field results in a
$\Gamma_h h\cos\psi$ term which additionally modifies the CPR and is
tunable by magnetic field. Figure 2(a) shows the effective
CPRs $\gamma(\psi)$ of a 0-π JJ for several different values of
magnetic field h. At $h = 0$ we might have a doubly degen-
erate ground state $\psi = \pm\varphi$ which, upon application of h,
transforms into a single one.
   The Josephson energy corresponding to CPR (18) is

$$U(\psi) = \langle j_c\rangle\left[1 - \cos\psi + \Gamma_h h\sin\psi + \frac{\Gamma_0}{2}\sin^2\psi\right]. \qquad (19)$$

The plots of $U(\psi)$ for the same set of h values is shown in
Fig. 2(b). Following the evolution of the curves at different
h one can see how the doubly degenerate ground state
transforms into a single one. Moreover, one can see that
at nonzero h the potential energy $U(\psi)$ lacks reflection
symmetry and therefore can be used to build Josephson
phase ratchets [22,23].
   The effective CPR (18) at $h = 0$ allows to calculate the
domain of existence of nontrivial solutions, which is de-
fined as $|\Gamma_0| > 1$ [18]. This results in

$$L_\pi \geq L_0\sqrt{\frac{3}{4L_0^2 + 3}}; \qquad L_0 \geq L_\pi\sqrt{\frac{3}{4L_\pi^2 + 3}}. \qquad (20)$$
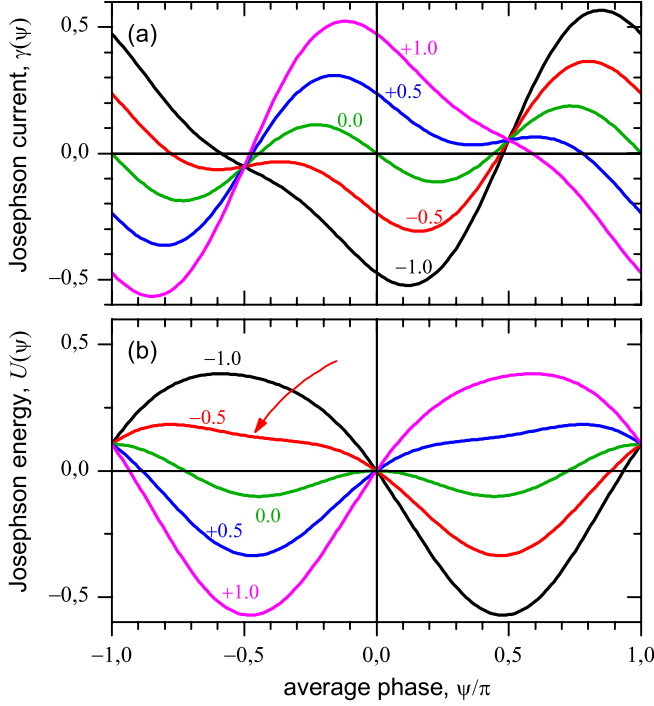
FIG. 2 (color online).   (a) effective CPR $\gamma(\psi)$ and (b) effective
Josephson energy $U(\psi)$ of a 0-π JJ with $L_0 = 1$, $L_\pi = 0.9$ for
several different values of magnetic field h given next to each
curve. The arrow in (b) points to a local energy minimum, which
appears at certain values of bias current.

Since we have made certain approximations, namely, used
Eq. (13) instead of Eq. (12), it is worth to compare the
domain (20) with the exact result [24], which reads

$$L_\pi \geq \arctan[\tanh(L_0)]; \tag{21a}$$
$$L_0 \geq \arctan[\tanh(L_\pi)], \tag{21b}$$

for our case $j_c^0 = j_c^\pi$. Both boundaries, the exact one given
by Eq. (21) and the approximate one given by Eq. (20) are
shown in Fig. 3. It is seen that our approximation works
extremely well not only for small L_0 and L_π, but also in the
limits $L_0, L_\pi \to \infty$, where it deviates from the asymptotic
value by only ~10%.

In the JJ with the CPR (18) and energy (19), one may
have several stable static solutions and several critical
(depinning) currents corresponding to the escape of the
phase from the relevant energy minimum. To find the
critical currents for a given magnetic field, we look for
extrema of the CPR (18) with respect to ψ. We arrive to

$$\cos\psi - \Gamma_h h \sin\psi - \Gamma_0(2\cos^2\psi - 1) = 0. \tag{22}$$

This problem can be reduced to a solution of a fourth order
polynomial, i.e., all solutions of Eq. (22) can be found
numerically. As a result we obtain up to four relevant roots
and up to four corresponding critical currents.

Several examples of $\gamma_c(h)$ dependences are shown in
Fig. 4. One can see that for the parameter set (a), which
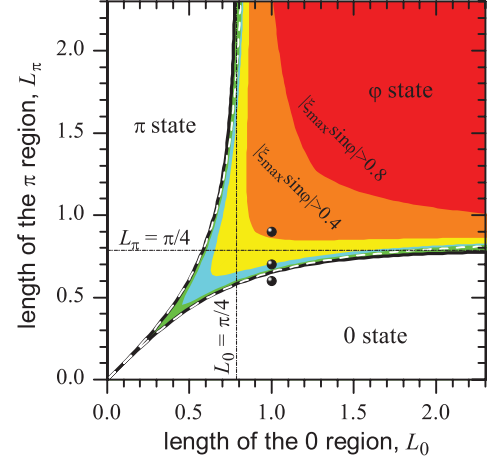corresponds to a state deep in the φ region at $h = 0$, see

FIG. 3 (color).   The domain of existence of a φ state ($h = 0$) on
the L_0, L_π plane. The exact boundary (21) (continuous lines) and
our approximation (20) (dashed lines) agree very well. The φ
domain has regions of different colors from green (approxima-
tion $|\xi(x) \sin\psi| \ll 1$ works well) to red (approximation is in-
valid). The boundaries correspond to $|\xi_{max} \sin\varphi| = 0.1, 0.2, 0.4$,
0.8. The dots indicate parameters used for discussion of Fig. 4.
Vertical and horizontal dash-dotted lines show asymptotic be-
havior of exact boundary (21).

Fig. 3, one observes a characteristic rotated diamondlike
shape with four critical currents in total for $|h| \lesssim 0.6$. The
two branches that meet at $h \approx +0.6 (-0.6)$ correspond to
the escape from the left (right) potential well in Fig. 2(b).
The upper (lower) branch corresponds to the escape in the
right (left) direction. At $|h| \approx 0.6$ the branches meet in-
dicating the disappearance of the corresponding local en-
ergy minimum. The two critical currents $\gamma_{c\pm}$ (in each
direction) observed at $h = 0$ coincide with $\gamma_\pm$ obtained
in Ref. [18]. One may be able to observe the lower one
experimentally in a system with low enough damping
using a special sweep sequence [18]. For some values of
h the smaller minimum can even be absent at $\gamma = 0$ [see
the arrow Fig. 2(b)] and appear at larger $|\gamma|$ [the vertical
dotted line corresponding to $h = 0.5$ in Fig. 4(a)]. The
small energy minimum appears for $0.03 < \gamma < 0.06$,
which also be seen in Fig. 2(b). Another distinct feature,
which might be measurable in experiment, is the shift of
the main minimum to $h = \pm 0.11$ in Fig. 4(a). This shift is
the evidence of a nontrivial phase state in the junction.
Finally, at large $|h|$ the upper branches of $\gamma_c(h)$ approach
the asymptotic lines

$$\gamma_c^{as}(h) \approx \pm\langle j_c\rangle\Gamma_h h, \tag{23}$$

which are also shown in Fig. 4. We remind that our analysis
is valid only for small values of $|h|$ and does not reproduce
the global $\gamma_c(h)$ features, presented elsewhere.

When the 0-π JJ gets more asymmetric, see Figs. 4(b)
and 4(c), the lower $\gamma_c(h)$ diamond-shaped domain be-
comes thinner and finally collapses. The point of collapse
corresponds to the crossing of the φ-domain boundary in
Fig. 3. Looking at $\gamma_c(h)$ one can see that the main cusplike
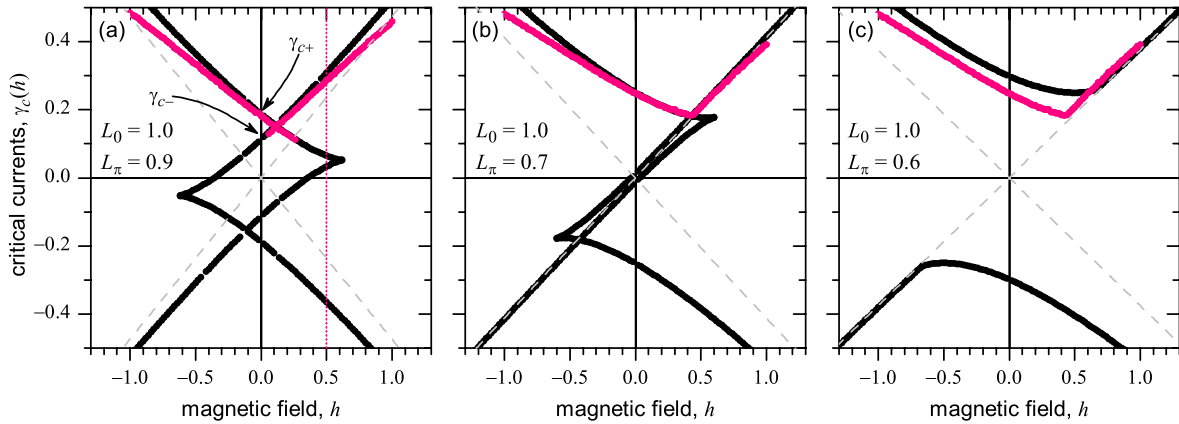
FIG. 4 (color online).   The critical current $\gamma_c(h)$ for 0-π JJs with different segment lengths. Dashed lines show the asymptotic behavior (23) for large $|h|$. Gray (pink) solid lines show the results of direct numerical simulations.

minimum shifts with asymmetry which can be used to extract the asymmetry from experimental data.

We also have performed numerical simulations of the $\gamma_c(h)$ dependence using STKJJ software [25]. The above analytical results coincide with the results of direct numerical simulation, shown by pink (gray) in Fig. 4, with good accuracy. This is especially so at $h \ll 1$ when the approximation $|\xi(x) \sin \psi| \ll 1$, used to derive (10), is valid.

To summarize, we have shown that the effective CPR (18) includes the term $\propto H \cos \psi$, which allows to tune the ground state. Thus, our system is a hybrid between φ [16] and φ_0 [26] JJs introduced earlier. The corresponding Josephson energy profile can be made asymmetric allowing to build Josephson phase ratchets. We have clarified how the $I_c(H)$ dependence of asymmetric 0-π JJ looks like for low magnetic field. It does have a minimum, but the minimum is shifted to some $\pm h_{\min}$ for positive and negative bias, respectively, so that the $I_c(H)$ curve is point symmetric.

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