Experimental Confirmation that the Proton is Asymptotically a Black Disk

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(Received 8 September 2011; published 15 November 2011)

Although experimentally accessible energies cannot probe "asymptopia", recent measurements of inelastic pp cross sections at the LHC at 7000 GeV and by Auger at 57 000 GeV allow us to conclude that (i) both $\sigma_{\rm inel}$ and $\sigma_{\rm tot}$, the inelastic and total cross sections for pp and $\bar{p}p$ interactions, saturate the Froissart bound of $\ln^2 s$, (ii) when $s \to \infty$, the ratio $\sigma_{\rm inel}/\sigma_{\rm tot}$ is experimentally determined to be 0.509 ± 0.021, consistent with the value 0.5 required by a black disk at infinite energies, and (iii) when $s \to \infty$, the forward scattering amplitude becomes purely imaginary, another requirement for the proton to become a totally absorbing black disk. Experimental verification of the hypotheses of analyticity and unitarity over the center of mass energy range $6 \le \sqrt{s} \le 57000$ GeV are discussed. In QCD, the black disk is naturally made of gluons; our results suggest that the lowest-lying glueball mass is 2.97 ± 0.03 GeV.

DOI: 10.1103/PhysRevLett.107.212002 PACS numbers: 12.38.Qk, 13.85.Hd, 13.85.Lg, 13.85.Tp

Introduction.—We discuss the implication of three new measurements of the high energy pp inelastic cross sections, $\sigma_{\rm inel}(\sqrt{s})$, where $\sqrt{s}$ is the center of mass (c.m.s.) energy. At $\sqrt{s} = 7000$ GeV, the Atlas collaboration [1] reports $\sigma_{\rm inel} = 69.4 \pm 2.4(\text{expt.}) \pm 6.9(\text{extr.})$ mb, with (expt.) and (extr.) the total experimental and extrapolation errors. The CMS collaboration [2], using a completely different technique, measures $\sigma_{\rm inel} = 68.0 \pm 2.0(\text{syst}) \pm 2.4(\text{lum.}) \pm 4(\text{extr.})$ mb, where (syst.) is the systematic error, (lum.) the error in luminosity and (extr.) is the extrapolation error for missing single and double diffraction events. Most recently, the Pierre Auger Observatory collaboration [3] reported a measurement of $\sigma_{\rm inel}^{p\text{-air}}$, the inelastic p-air cross section at $\sqrt{s} = 57\,000 \pm 6000$ GeV. This measurement, after correction for a 25% He^4 contamination in a cosmic ray beam consisting mostly of protons at that energy, was converted by a Glauber calculation into the pp inelastic cross section [3], $\sigma_{\rm inel} = 90 \pm 7(\text{stat}) \pm^{9}_{11} (\text{syst}) \pm 1.5(\text{Glaub.})$, with the statistical (stat), the systematic (syst) errors, and the estimated error in the Glauber (Glaub.) calculation. With a cosmic ray measurement at 57000 GeV it is likely that we are now experimentally as close to asymptopia (defined here as the energy behavior of hadron-proton cross sections near $s \to \infty$) as we will ever get.

Block and Halzen (BH) [4,5] have made an analyticity constrained amplitude fit to lower energy data ($6 \le \sqrt{s} \le 2000$ GeV) that shows that $\sigma_{\rm tot}$ for $\bar{p}p$ and pp asymptotically saturates the Froissart bound [6]. This note exploits the new higher energy measurements of $\sigma_{\rm inel}$ in order to make accurate predictions at asymptotia based only on measurements of pp and $\bar{p}p$ cross sections in the energy range $6 \le \sqrt{s} \le 57\,000$ GeV. While the analyticity constrained amplitude model of BH [4,5] yields the total cross sections and the ρ value, the ratio of real and the imaginary parts of the forward scattering amplitude, an eikonal model, dubbed the "Aspen" model [7], will be used to obtain the ratio of the inelastic to total cross sections, $r(\sqrt{s}) \equiv \sigma_{\rm inel}(\sqrt{s})/\sigma_{\rm tot}(\sqrt{s})$. We will show that the resulting ρ value and the ratio of $\sigma_{\rm inel}/\sigma_{\rm tot}$ at $\sqrt{s} = \infty$ are consistent with the proton being an expanding black disk, presumably of gluons; our fits to $\sigma_{\rm inel}$ and $\sigma_{\rm el}$ will allow us to infer a lowest-lying glueball mass of 2.97 ± 0.03 GeV. Furthermore, we will show that both the Martin-Froissart bound [6,8] on the pp and $\bar{p}p$ total cross sections and the Martin bound [9] on the pp and $\bar{p}p$ inelastic cross sections are saturated, from $6 \le \sqrt{s} \le 57\,000$ GeV.

The analytic amplitude model.—Using this approach, BH was able to claim accurate predictions of the forward pp ($\bar{p}p$) scattering properties, $\sigma_{\rm tot} \equiv \frac{4\pi}{p}\,{\rm Im} f(\theta_L = 0)$ and $\rho \equiv \frac{{\rm Re} f(\theta_L=0)}{{\rm Im} f(\theta_L=0)}$, using the analyticity-constrained analytic amplitude model [5] that saturates the Froissart bound [6]; here $f(\theta_L)$ is the pp laboratory scattering amplitude with θ_L, the laboratory scattering angle and p is the laboratory momentum. By saturation of the Froissart bound, we mean that the total cross section $\sigma_{\rm tot}$ rises as $\ln^2 s$. Furthermore, the use of analyticity constraints allows one to anchor fits at 6 GeV to the very accurate low energy cross section measurements between 4 and 6 GeV in the spirit of finite energy sum rules (FESR) [10]. A local fit is made of the experimental values of $\sigma^{\pm}$ between 4 and 6 GeV, for both $\bar{p}p$ and pp, from which BH [5] derive precise 6 GeV anchor points for $\sigma^{\pm}$ and their energy derivatives in Eq. (1). The results are actually consistent with those obtained with old-fashioned FESR [11]. The model parameterizes the even and odd (under crossing) cross sections and fits [5] four experimental quantities, $\sigma_{\bar{p}p}(\nu)$, $\sigma_{pp}(\nu)$, $\rho_{\bar{p}p}(\nu)$ and $\rho_{pp}(\nu)$ to the high energy parameterizations

$$\sigma^{\pm}(\nu) = \sigma^0(\nu) \pm \delta\left(\frac{\nu}{m}\right)^{\alpha-1}, \quad (1)$$

$$\rho^{\pm}(\nu) = \frac{1}{\sigma^{\pm}(\nu)}\left\{\frac{\pi}{2}c_1 + c_2\pi\ln\left(\frac{\nu}{m}\right) - \beta_{\mathcal{P}'}\cot\left(\frac{\pi\mu}{2}\right)\left(\frac{\nu}{m}\right)^{\mu-1} + \frac{4\pi}{\nu}f_+(0) \pm \delta\tan\left(\frac{\pi\alpha}{2}\right)\left(\frac{\nu}{m}\right)^{\alpha-1}\right\}, \quad (2)$$

where the upper sign is for pp and the lower sign is for $\bar{p}p$, and, for high energies, $\nu/m \simeq s/2m^2$. Here the even amplitude cross section σ^0 is given by

$$\sigma^0(\nu) \equiv \beta_{\mathcal{P}'}\left(\frac{\nu}{m}\right)^{\mu-1} + c_0 + c_1\ln\left(\frac{\nu}{m}\right) + c_2\ln^2\left(\frac{\nu}{m}\right), \quad (3)$$

where ν is the laboratory energy of the incoming proton (antiproton), m the proton mass, and the "Regge intercept" $\mu = 0.5$. The predictions for the pp and $\bar{p}p$ total cross sections are shown in Fig. 1. The dominant $\ln^2(s)$ term in the total cross section [Eq. (3)] saturates the Froissart bound [6]; it controls the asymptotic behavior of the cross sections. BH made a simultaneous fit [5] to the pp and $\bar{p}p$ data for the ρ value, the ratio of the real to the imaginary forward scattering amplitudes, shown in Fig. 2. From Eqs. (2) and (3), we see that in the limit of $s \to \infty$, $\rho \to 0$ as $1/\ln s$, (albeit very slowly), a necessary condition for a black disk. Although the ρ-values are essentially the same for $\bar{p}p$ and pp for $\sqrt{s} > 100$ GeV, at the highest accelerator energies, ρ only changes from 0.135 at 7000 GeV to 0.132 at 14000 GeV. Clearly, we are no where near asymptopia, where $\rho = 0$.

With two low energy constraints at 6 GeV and 4 parameters, precise values for c_0 and $\beta_{\mathcal{P}'}$ could be obtained [5]. The fitted values for the coefficients of $\sigma^0(\nu)$ of Eq. (3) for the fit for $6 \leq \sqrt{s} \leq 2000$ GeV are listed in Table I. Evaluating Eq. (3) at 57 000 GeV, we predict $\sigma_{\text{tot}} = 134.8 \pm 1.5$ mb for pp interactions. We note that c_2, the coefficient of $\ln^2(s)$, is well determined, having a statistical accuracy of ~2%. Thus, experimental data show that the Froissart bound is satisfied for total cross sections σ_{tot} for both $\bar{p}p$ and for pp in the energy interval $6 \leq \sqrt{s} \leq 2000$ GeV.

Aspen model.—The Aspen model [7] is an eikonal model that describes experimental $\bar{p}p$ and pp data for σ_{tot}, ρ and the slope parameter $B \equiv d[\ln d\sigma_{\text{el}}/dt]_{t=0}$, the logarithmic derivative of the forward differential elastic scattering cross section, where t is the square of the 4-momentum transfer. Among many other quantities, it allows one to accurately predict the ratio $r = \sigma_{\text{el}}(\nu)/\sigma_{\text{tot}}(\nu)$, i.e., the ratio of the elastic to total cross section for both $\bar{p}p$ and pp, as a function of energy, where again, the total cross sections have been anchored at 6 GeV by FESR constraints [10]. Details of the model are given in Ref. [4,7]. As is the case of the total cross sections, the values for r are essentially identical for $\bar{p}p$ and pp for c.m.s. energies $\sqrt{s} \geq 100$ GeV. The ratio r is plotted in Fig. 3. Again, we see that we are far from asymptopia, where the black

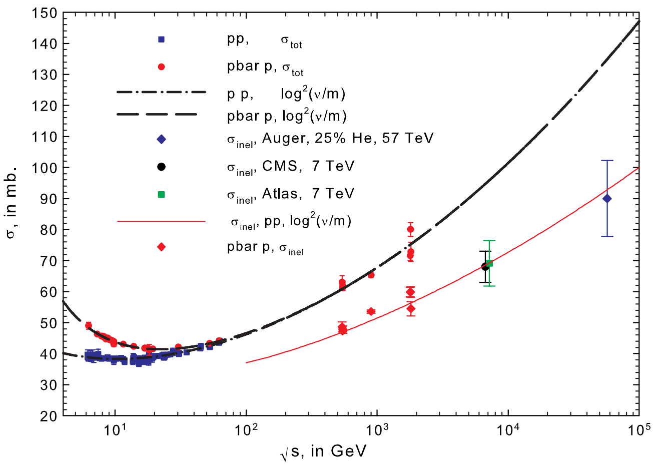

FIG. 1 (color online). The fitted total cross section, σ_{tot}, for $\bar{p}p$ (dashed curve) and pp (dot-dashed curve) from Eq. (1), in mb vs $\sqrt{s}$, the c.m.s. energy in GeV, taken from BH [5]. The $\bar{p}p$ data used in the fit are the (red) circles and the pp data are the (blue) squares. The fitted data were anchored by values of $\sigma_{\text{tot}}^{\bar{p}p}$ and σ_{tot}^{pp}, together with the energy derivatives $d\sigma_{\text{tot}}^{\bar{p}p}/d\nu$ and $d\sigma_{\text{tot}}^{pp}/d\nu$ at 6 GeV using FESR, as described in Ref. [5]. The lowest (red) solid curve that starts at 100 GeV is our *predicted* inelastic cross section from Eq. (5), σ_{inel}, in mb, vs $\sqrt{s}$, in GeV. The lowest energy inelastic data, the $\bar{p}p$ (red) diamonds, were *not* used in the fit, nor were the 3 high energy pp inelastic measurements, the (black) circle CMS value, the (green) square Atlas measurement and the (blue) diamond Auger measurement. As clearly seen, our inelastic prediction from Eq. (5), which also asymptotically behaves as $\ln^2(s)$, is in excellent agreement with the new measurements of the inelastic cross section at very high energy.

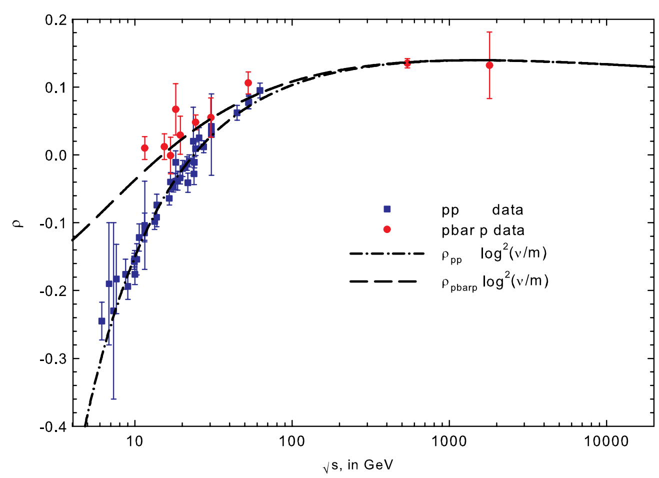

FIG. 2 (color online). The fitted ρ-value, for $\bar{p}p$ (dashed curve) and pp (dot-dashed curve) from Eq. (1) vs $\sqrt{s}$, the c.m.s. energy in GeV. The $\bar{p}p$ data used in the fit are the (red) circles and the pp data are the (blue) squares.

TABLE I. Values of the parameters for the even amplitude, $\sigma^0(\nu)$, using 4 FESR analyticity constraints (taken from Ref. [5]).

$c_0 = 37.32$ mb,	$c_1 = -1.440 \pm 0.070$ mb,	$c_2 = 0.2817 \pm 0.0064$ mb,	$\beta_{\mathcal{P}'} = 37.10$ mb

disk model implies a ratio $r = 1/2$, whereas at 57 000 GeV, we predict $r \sim 0.32$.

Predicting the inelastic cross section.—We are now ready to predict $\sigma_{\rm inel}(\nu) \equiv (1 - r(\nu))\sigma^0(\nu)$ *numerically* for $\sqrt{s} \geq 100$ GeV, using $r(\nu)$ obtained above, together with the fitted even amplitude cross section $\sigma^0(\nu)$ of Eq. (3) determined by the parameters of Table I. We emphasize that our prediction of $\sigma_{\rm inel}$ does not use any inelastic scattering data. Since the approach is at this point purely numerical, we decided to fit the inelastic numbers with the same analytical parameterization as was used for the total cross section $\sigma^0(\nu)$ in Eq. (3). The analytic expression for our prediction of the even amplitude high energy inelastic cross section $\sigma^0_{\rm inel}(\nu)$ given by

$$\sigma^0_{\rm inel}(\nu) \equiv \beta^{\rm inel}_{\mathcal{P}'}\left(\frac{\nu}{m}\right)^{\mu-1} + c_0^{\rm inel} + c_1^{\rm inel}\ln\left(\frac{\nu}{m}\right) + c_2^{\rm inel}\ln^2\left(\frac{\nu}{m}\right) \tag{4}$$

$$= 62.59\left(\frac{\nu}{m}\right)^{-0.5} + 24.09 + 0.1604\ln\left(\frac{\nu}{m}\right) + 0.1433\ln^2\left(\frac{\nu}{m}\right)\ \text{mb} \tag{5}$$

accurately reproduces the numerical values of $\sigma_{\rm inel}(\nu)$ to better than 4 parts in 10^4 over the energy range $100 \leq \sqrt{s} \leq 100\,000$ GeV. This new result for $\sigma^0_{\rm inel}(\nu)$ implies that the Froissart bound is also saturated for high energy *inelastic* cross sections in the energy interval $100 \leq \sqrt{s} \leq 57\,000$ GeV, a result anticipated theoretically by Andre Martin [9], using analyticity and unitarity. Figure 1 shows that our $\ln^2(s)$ prediction of Eq. (5) for $\sigma^0_{\rm inel}(\nu)$, the lower (red) solid curve, is in excellent agreement with all experimental data, up to the highest possible energy. The (red) diamonds, are $\bar{p}p$ inelastic cross sections. The LHC 7000 GeV pp inelastic cross section data points are the (black) circle from CMS [2] and the (green) square from Atlas [1], slightly separated for visual purposes. The (blue) diamond is the Auger inelastic cross section [3] for a 25% He^4 contamination of their $\sigma_{\rm in}^{p\text{-air}}$ cross section at 57 000 GeV. We reiterate that none of these experimental inelastic cross sections were used in our fits that predicted high energy inelastic cross sections; our predictions at 7000 GeV are $\sigma_{\rm inel} = 69.0 \pm 1.3$ mb and at 57 000 GeV, $\sigma_{\rm inel} = 92.9 \pm 1.6$ mb.

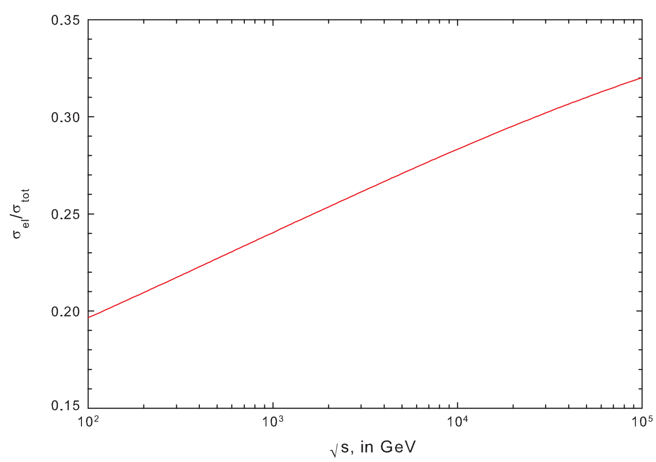

FIG. 3 (color online). The r value, the ratio of $\sigma_{\rm el}/\sigma_{\rm tot}$, vs $\sqrt{s}$, the c.m.s. energy in GeV.

Evidence for a black disk.—It is unlikely that there will ever be higher energy measurements for $\sigma_{\rm inel}$ for either $\bar{p}p$ or pp collisions, yet our results show that present measurements are far from asymptopia. Nevertheless, the data give us a consistent picture of asymptopia by the compelling evidence that both the elastic and inelastic cross sections saturate the Froissart bound. The addition of the inelastic cross section of Eq. (5) going as $\ln^2 s$ now allows us to explore asymptopia *experimentally*; we find the limit of $\sigma_{\rm inel}(s)/\sigma_{\rm tot}(s)$ as $s \to \infty$ simply by taking the ratio of the $\ln^2(s)$ terms in Eq. (5) and (3). We find the experimentally determined value at infinity,

$$\lim_{s\to\infty}\frac{\sigma_{\rm inel}(s)}{\sigma_{\rm tot}(s)} = \frac{c_2^{\rm inel}}{c_2} = 0.509 \pm 0.011, \tag{6}$$

a result compatible with the ratio 1/2 predicted for a black disk. Satisfying this ratio of the inelastic to the total cross section at infinity gives us the first experimental evidence that the proton becomes an expanding black disk at asymptopia. We have already shown that the second condition, $\rho = 0$, i.e., the amplitude is imaginary, is also satisfied. The model of Troshin [12] in which the elastic scattering dominates over the inelastic is thus falsified, whereas the models [13,14] in which the proton becomes a black disk asymptotically are now justified experimentally.

Properties of a black disk.—In impact parameter space b, the elastic and total cross sections are given by

$$\sigma_{\rm el} = 4\int d^2b|a(b,s)|^2, \quad \sigma_{\rm tot} = 4\int d^2b\,{\rm Im}\,a(b,s). \tag{7}$$

The amplitude $a(s,b)$ of the black disk of radius R is given by

$$a(b,s) = \frac{i}{2}, \qquad 0 \leq b \leq R,$$
$$a(b,s) = 0, \qquad b > R, \tag{8}$$

so that (for details, see Ref. [15])

$$\sigma_{\rm tot} = 2\pi R^2, \qquad \sigma_{\rm inel} = \sigma_{\rm el} = \pi R^2,$$
$$\frac{\sigma_{\rm inel}}{\sigma_{\rm tot}} = 0.5, \qquad \frac{d\sigma_{\rm el}}{dt} = \pi R^4\left[\frac{J_1(qR)}{qR}\right]^2, \tag{9}$$

where $q^2 = -t$.

Magnitude of the $\ln^2 s$ *coefficient of the bound.*—Using analyticity and unitarity, Andre Martin has recently found a more rigorous *inelastic* hadron-proton bound [9], using $|t| = (2m_\pi)^2$, i.e.,

$$\sigma_{\rm inel} < \frac{\pi}{4m_\pi^2}\ln^2 s, \quad \text{so that } \sigma_{\rm tot} < \frac{\pi}{2m_\pi^2}\ln^2 s \tag{10}$$

where for the total cross section bound we have invoked the black disk ratio of 2 to 1. The use of m_π in the two-particle mass $M = 2m_\pi$ is clearly wrong experimentally, since $\frac{\pi}{2m_\pi^2}\ln^2(\nu/m) = 31.23\ln^2(\nu/m)$ mb, whereas experimentally we have obtained $c_2\ln^2(\nu/m) = 0.2817\ln^2(\nu/m)$ mb, a cross section 2 orders of magnitude smaller, implying that the scale is not set by the pion mass but by a mass scale 1 order of magnitude larger. Reinterpreting $M = 2m_\pi$ in Eq. (10) as the lowest-lying glueball mass which we call $M_{\rm glueball}$, we find $M_{\rm glueball} = (2\pi/c_2)^{1/2} = 2.97 \pm 0.03$ GeV. Recent fully relativistic and crossing symmetric AdS/QCD theories [16–18] may provide a link for our mass reinterpretation, as well as providing new constraints on gravity. Finally, we note that lattice QCD using only gluons [19] predicts the lowest-lying 1^{+-} glueball state at $M_G = 2.940 \pm 0.140$ GeV, a result in tantalizingly close agreement with our new mass scale. Obviously, the definition of our new scale is still arguable. Further, if the asymptotic proton is a black disk of gluons, the high energy behavior is flavor blind and the coefficient of the $\ln^2 s$ term is the same for all reactions, from πp to γp scattering. Support for this claim comes from both the COMPETE group [20] and Ishida and Igi [21].

Conclusions.—We find that the $\ln^2 s$ Froissart bounds for the proton for both $\sigma_{\rm tot}$ [6] and $\sigma_{\rm inel}$ [9] are saturated, allowing us to determine at infinite s that: (i) the experimental ratio $\sigma_{\rm inel}/\sigma_{\rm tot} = 0.509 \pm 0.011$, compatible with the black disk ratio of 0.5 and (ii) the forward scattering amplitude is purely imaginary. We thus conclude that the proton becomes an expanding black disk at sufficiently ultrahigh energies that are probably never accessible to experiment. The theory for these bounds is predicated on the pillar stones of analyticity and unitarity, which have now been experimentally verified up to 57 000 GeV. Further, since $\sigma_{\rm tot}$ has been extrapolated up from the Tevatron, we expect no new large cross section physics between 2000 and 57 000 GeV. Finally, we infer that the lowest-lying glueball mass is at $M_{\rm glueball} = 2.97 \pm 0.03$ GeV, very close to the lattice QCD value [19] of the lowest-lying 1^{+-} state.

In part, F. H. is supported by the National Science Foundation Grant No. OPP-0236449, by the DOE grant DE-FG02-95ER40896 and by the University of Wisconsin Alumni Research Foundation. M. M. B. thanks the Aspen Center for Physics, supported in part by NSF Grant No. 1066293, for its hospitality during this work.

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