Topological Order at Nonzero Temperature

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(Received 30 June 2011; published 18 November 2011)

We propose a definition for topological order at nonzero temperature in analogy to the usual zero temperature definition that a state is topologically ordered, or ''nontrivial'', if it cannot be transformed into a product state (or a state close to a product state) using a local (or approximately local) quantum circuit. We prove that any two-dimensional Hamiltonian which is a sum of commuting local terms is not topologically ordered at $T > 0$. We show that such trivial states cannot be used to store quantum information using certain stringlike operators. This definition is not too restrictive, however, as the four dimensional toric code does have a nontrivial phase at nonzero temperature.

DOI: 10.1103/PhysRevLett.107.210501 PACS numbers: 03.67.Mn, 05.30.Pr, 71.10.Pm

Topological quantum computation [1,2] is one of the most promising approaches for building a quantum computer. The nonlocal encoding of quantum information in the ground state subspace of topologically ordered systems protects it against decoherence. Topological order is also one of the most interesting current topics in condensed matter physics, as models such as the Levin-Wen models [3] provide a method for constructing novel phases.

One way of characterizing the nontrivial properties of these phases is the following definition: a state is ''trivial'' if it is a product state, or if it can be transformed into a state which approximates a product state using a local or approximately local quantum circuit. A state is nontrivial, or topologically ordered, otherwise. Despite the usefulness of this definition in terms of circuits, we do not have a similarly useful definition for states at temperature $T > 0$.

Here, we propose an analogous ''circuit definition'' for topological order at $T > 0$. Another method used previously to study topological order at $T > 0$ is topological mutual information [4,5], a generalization of topological entanglement entropy [6,7]. While a nonzero topological mutual information for T near zero is a useful numerical signature of nontrivial phases [8], the use of topological entropy can give different answers from the circuit definition even for at $T = 0$ [9], so both definitions are useful to fully characterize a state. Also, it has been shown that in many two-dimensional theories there exist stringlike logical operators [10,11]. If these logical operators can be generated by dragging defects with energy $O(1)$, then, since there is a nonzero density of such defects at $T > 0$, the information which would be topologically protected at zero temperature will decohere rapidly at $T > 0$ (though we emphasize that the defect density is exponentially small in $1/T$; as an experimental example, fractional Hall conductance is accurately quantized [12] even at $T > 0$). While this kind of ''operational definition'' is practically useful, the circuit definition here will lead to similar operational results for *all* Hamiltonians which are a sum of local commuting terms.

Conversely, we will show that the four dimensional toric code [13] has topological order under the circuit definition for sufficiently small $T > 0$ (this will not be a rigorous proof since we will heuristically argue for the existence of certain operators using previous results, but we then prove that the existence of these operators is inconsistent with a trivial state). One advantage of a definition using the density matrix rather than the excitation above the ground state is seen in a toy system in the Supplemental Material which is topologically trivial at $T = 0$ but nontrivial at some small $T > 0$.

Topological order at zero temperature.—We start with various previous definitions of topological order at $T = 0$. Topologically ordered Hamiltonians according to one definition have a ground state degeneracy that depends upon the topology of the manifold, and the ground state subspace obeys a property called the ''disk axiom'' or ''TQO-1'' [1,2,14–16], which we quantify by:

Definition 1.—Let L denote the system size and let P denote the projector onto the ground state subspace. Let L^* denote some length smaller than L. Then, the ground state subspace has (L^*, ϵ) topological degeneracy [17] if for any operator O supported on a set of diameter smaller than L^* there is a scalar z such that

$$\|POP - zP\| \leq \epsilon. \tag{1}$$

Using circuits we can also define topological order for systems with a unique ground state. Consider a unitary quantum circuit U where the depth of the circuit multiplied by the maximum range of each unitary in the circuit is bounded by some range R. Note that for any operator O supported on a set Z, $U^\dagger OU$ is supported on the set of sites within distance R of Z.

Definition 2.—Let ψ_0 be the ground state of the Hamiltonian H. We say that the state ψ_0 is (R, ϵ) trivial

if there exists a unitary quantum circuit U with range R such that $|\psi_0 - U\psi_{\text{prod}}| \leq \epsilon$ for some product state ψ_{prod}.

Certainly, every state is $(L, 0)$ trivial, so we are only interested in the case of $R < L$. Colloquiually speaking a system will be trivial if it is (R, ϵ) trivial for some $\epsilon \ll 1$ and for some $R \ll L$. We can relate the two definitions [17]: if a system has (L^*, ϵ) topological degeneracy, then no state ψ_0 in the ground state subspace is (R, δ) trivial for $R < L^*/2$ and sufficiently small δ, ϵ. To see this, suppose ψ_0 is (R, δ) trivial. Then, the expectation of any operator O in state ψ_{prod} is close to the expectation of $\langle\psi_0|UOU^\dagger|\psi_0\rangle$. If O is supported on a single site, $UOU^\dagger$ is supported on a set of diameter less than L^* and so for such O, $\langle\psi_{\text{prod}}|O|\psi_{\text{prod}}\rangle \approx \langle U^\dagger\psi_1|O|\psi^\dagger\psi_1\rangle$. That is, the subspace spanned by ψ_{prod} and $U^\dagger\psi_1$ has $(1, \eta)$ topological degeneracy for $\eta = \mathcal{O}(\epsilon + \delta)$. However, for η sufficiently small compared to inverse system size, no such state $U^\dagger\psi_1$ exists since ψ_{prod} is a product state.

Topological order at nonzero temperature.—We begin by defining a "classical state of range R" ρ_{cl} (this will replace the use of a product state when we define $T > 0$ topological order) to be a state such that $\rho_{\text{cl}} = Z^{-1}\exp(-H_{\text{cl}})$ where Z is a normalization factor and H_{cl} is a Hamiltonian which is a sum of terms all acting on sets of diameter at most R and all of which are diagonal in a product basis.

Definition 3.—A density matrix ρ is (R, ϵ) trivial if it is possible to tensor in additional degrees of freedom $\mathcal{K}_i$ on each site, defining an enlarged space with Hilbert space $\mathcal{H}_i \otimes \mathcal{K}_i$ on each site, such that

$$|\rho - \text{Tr}_{\{\mathcal{K}_i\}}(U\rho_{\text{cl}}U^\dagger)| \leq \epsilon, \tag{2}$$

where the $|\ldots|$ denotes the trace norm (the trace norm of a Hermitian operator is the sum of the absolute values of its eigenvalues), where U is a unitary quantum circuit with range R, and ρ_{cl} is a classical state of range R (both U and ρ_{cl} are defined on the enlarged space), and where the trace is over the added degrees of freedom $\mathcal{K}_i$.

If we allow H_{cl} to be unbounded and we allow the dimension of $\mathcal{K}_i$ to be unbounded, then this definition is equivalent to saying that ρ is (R, ϵ) trivial if it is, up to error ϵ in trace norm, equal to an incoherent sum of $(R, 0)$ trivial states: $\rho = \sum_a P(a)|\psi_{\text{triv}}(a)\rangle\langle\psi_{\text{triv}}(a)|$, for some probability distribution $P(a)$. However, for some purposes one might want to construct ρ using a bounded H_{cl} and a bounded dimension on $\mathcal{K}_i$.

Absence of topological order for two-dimensional Hamiltonians with commuting terms.—We now show absence of topological order for any $T > 0$, under the above definition, for two-dimensional Hamiltonians which are a sum of commuting projectors. The proof is based on showing that the density matrix can be approximately written as a weighted sum over density matrices of a system with "holes" in it as explained below, and then using results from [18] to write each such density matrix as a trivial state.

Consider a two-dimensional Hamiltonian $H = \sum_X Q_X$, where the terms Q_X are commuting projectors. Assume that the terms in the Hamiltonian are local, so that each projector Q_X is supported on some set X which has diameter R_{int} which is $O(1)$. Further suppose that each site is in at most $O(1)$ of such sets X, and for simplicity consider a square lattice.

The density matrix is $\rho = Z^{-1}\exp(-\beta H)$, where Z is a normalization and $\beta = T^{-1}$. Note that for any projector Q_X we have $\exp(-\beta Q_X) = \sum_{s_X\in\{0,1\}}((1 - s_X) \times \exp(-\beta)I + s_X(1 - \exp(-\beta))(I - Q_X))$, where we introduce an additional variable s_X, and sum over $s_X = 0, 1$ (this variable s_X is unrelated to any local spin degrees of freedom of the Hamiltonian). Thus,

$$\rho = \sum_{\{s_X\}} \frac{Z(\{s_X\})}{Z}\rho(\{s_X\})P(\{s_X\}), \tag{3}$$

where the sum is over a set of variables s_X, each variable taking values 0 or 1, with $P(\{s_X\}) = \prod_X[(1 - s_X)\exp(-\beta) + s_X(1 - \exp(-\beta))]$, and where

$$\rho(\{s_X\}) \equiv Z(\{s_X\})^{-1}\prod_X((1 - s_X)I + s_X(I - Q_X)), \tag{4}$$

with $Z(\{s_X\})^{-1}$ being a normalization. Note that $\rho(\{s_X\})$ is maximally mixed on the ground state subspace of $H(\{s_X\}) \equiv \sum_X s_X Q_X$.

We will show that the density matrix ρ, for any given β, is dominated by a sum over choices of s_X in which there are lots of holes in the lattice, where a hole corresponds to a disk Y with radius greater than R_{int}, such that for any X with $X \cap Y \neq \emptyset$, we have $s_X = 0$. See Fig. 1. (Note that the

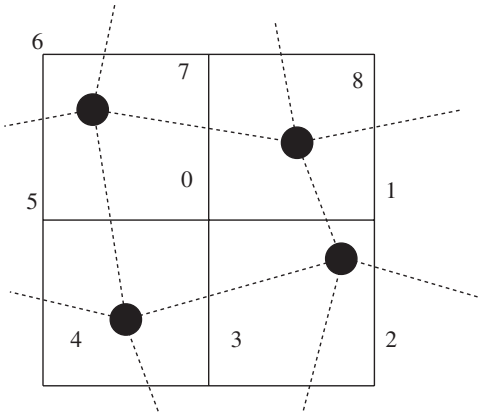

FIG. 1. Illustration of holes in the lattice. The solid lines indicate the division of the lattice into squares of size l_β, and only some of the squares are illustrated. Solid circles indicate holes in the lattice, with one hole per square. Dashed lines connect holes (dashed line extending outside the solid lines connecting to only one hole are intended to indicate connections to holes in other square which are not illustrated). Symbols 0, 1, 2, ... indicate different subsets of the lattice with holes in it, with the dashed lines indicating divisions between those subsets.

variables s_X for X not intersecting such a disk Y are not determined by the choice of holes, these variables may be either 0 or 1). We divide the square lattice into large squares of linear size l_β, with l_β exponentially large in β as given in Eq. (8) and logarithmically large in system size. We call a configuration "valid" if there is at least one hole per square. We will show that for valid configurations the density matrix $\rho(\{s_X\})$ can be expressed as a local unitary with range $R = 2l_\beta$ acting on a classical state and we bound the contribution to ρ from invalid configurations. Combining these results implies that the sum over s_X can be approximated by a local unitary acting on a classical state (the error ϵ arises from the invalid configurations of s_X).

Consider a given valid choice of s_X. For such a valid choice of s_X, we draw an image such as in Fig. 1. As mentioned, the terms s_X may be zero even if X does not intersect a hole. If a given choice of s_X has more than one hole per square, we only indicate one hole in the square, choosing which one to take according to any arbitrary rule (for example, the hole closest to the top left corner). We draw dashed lines connecting the holes. The dashed lines break the lattice outside the holes into regions labeled 0, 1, 2, ... as shown. Then, $H(\{s_X\})$ can be rewritten as

$$H(\{s_X\}) = \sum_{\langle a,b\rangle} H_{a,b} + \sum_a H_a, \tag{5}$$

where a and b label regions bounded by dashed lines, the sum is over neighboring a, b, and $H_{a,b}$ is supported on regions a and b (define $H_{a,b} = H_{b,a}$) and H_a is supported on region a.

So, by coarse graining the lattice, the Hamiltonian is a sum of commuting terms, each acting on at most two regions. For any a, the terms $H_{a,b}$ and H_a which are supported on that region all commute with each other. This allows us to find a decomposition of the Hilbert space which greatly simplifies the description of the problem. Equations (6) and (7), below differ slightly from lemma 8 of [18], but can be proven using the same tools of interaction algebra [19] (see also [20]): decompose $H_{a,b}$ as a sum of product operators $H_{a,b} = \sum_\gamma O_a^{ab}(\gamma) O_b^{ab}(\gamma)$, where the operators $O_a^{ab}(\gamma)$, $O_b^{ab}(\gamma)$ are supported on a, b respectively and the operators $O_b^{ab}(\gamma)$ are chosen from an orthonormal basis. Then, $[O_a^{ab}(\delta), O_a^{ac}(\gamma)] = 0$ for $b \neq c$, for all δ, γ. Let $\mathcal{A}^{ab}$ be the algebra generated by the set of $O_a^{ab}(\gamma)$ for given b. The algebras $\mathcal{A}^{ab}$, $\mathcal{A}^{ac}$ commute for $b \neq c$.

Let $\mathcal{H}_a$ denote the Hilbert space on region a. One way for the two algebras to commute is simply that $\mathcal{H}_a$ decomposes into a tensor product of Hilbert spaces, and each $\mathcal{A}^{ab}$ acts on a different space. However, this is not the only possibility. Suppose, for example, the Hamiltonians $H_{a,b}$ are all diagonal in some product basis. For example, consider an Ising Hamiltonian with all terms involving only operators S^z. Then all the $H_{a,b}$ would commute, but we would not have this tensor product decomposition. However, we can decompose $\mathcal{H}_a$ into a direct sum of Hilbert spaces $\mathcal{H}_a^{\alpha(a)}$, and then further decompose each such Hilbert space $\mathcal{H}_a^{\alpha(a)}$ into a tensor product of spaces $\mathcal{H}_{a\to b}^{\alpha(a)}$ giving

$$\mathcal{H}_a = \bigoplus_{\alpha(a)} \mathcal{H}_a^{\alpha(a)} = \bigoplus_{\alpha(a)} (\mathcal{H}_{a,a}^{\alpha(a)} \otimes \bigotimes_{\langle b,a\rangle} \mathcal{H}_{a\to b}^{\alpha(a)}), \tag{6}$$

where the product is over b that neighbor a, such that each operator $H_{a,b}$ can be decomposed as

$$H_{a,b} = \sum_{\alpha(a),\beta(b)} P_a^{\alpha(a)} P_b^{\beta(b)} H_{a,b}^{\alpha(a),\beta(b)}, \tag{7}$$

where $P_a^{\alpha(a)}$ is the operator on $\mathcal{H}_a$ which projects onto $\mathcal{H}_a^{\alpha(a)}$ and $H_{a,b}^{\alpha(a),\beta(b)}$ acts on the subspace of $\mathcal{H}_a^{\alpha(a)} \otimes \mathcal{H}_b^{\beta(b)}$ given by $\mathcal{H}_{a\to b}^{\alpha(a)} \otimes \mathcal{H}_{b\to a}^{\beta(b)}$.

Define $Q_{a,b}$ to project onto the nonzero energy states of $H_{a,b}$, so that the maximally mixed state on the ground state subspace of $H(\{s_X\}$ is $Z(\{s_X\})^{-1} \prod_{\langle a,b\rangle} (I - Q_{a,b}) \prod_a (I - Q_a)$, where $Z(\{s_X\})$ is a normalization factor. This state is a $(2, 0)$ trivial state on the coarse-grained lattice of regions because for any choice of the variables $\alpha(a)$ for each region a the projection of $\rho(\{s_X\})$ onto the product of spaces $\mathcal{H}_a^{\alpha(a)}$ is a product state on the spaces $\mathcal{H}_{a\to b}^{\alpha(a)} \otimes \mathcal{H}_{b\to a}^{\beta(b)}$. Thus it is an $(R, 0)$ trivial state on the original lattice for $R = 2l_\beta$.

We now bound the contribution of invalid configurations. Note that given any two sequences $\{s_X\}$ and $\{s'_X\}$ such that $s'_X \geq s_X$ for all X we have $Z(\{s'_X\}) \leq Z(\{s_X\})$. Imagine breaking each large square of linear size l_β into small squares of size $2R_{\text{int}}$ on each side (this will not give the best estimates but simplifies the proof), and let n_C be the maximum number of projectors that intersect any of those small squares. Then, the sum of $Z(\{s_X\})P(\{s_X\})$ over all $\{s_X\}$ such that there is a given configuration of holes in those small squares with a total of k such holes is at least equal to $\exp(-(2R_{\text{int}}^2 k\beta))$ times the sum of the same quantity over configurations with no holes. So, the sum of $Z^{-1}Z(\{s_X\})P(\{s_X\})$ over configurations with no holes in any given square is bounded by $(1 - \exp(-R_{\text{int}}^2\beta))^{(l_\beta/R_{\text{int}})^2}$, as follows by counting the number of configurations with holes and without.

Pick

$$l_\beta = \exp((2R_{\text{int}})^2\beta) R_{\text{int}} \log(V)/\log(\epsilon), \tag{8}$$

so $[1 - \exp(-(2R_{\text{int}})^2\beta)]^{(l_\beta/R_{\text{int}})^2} \leq \epsilon/V$. Then, the contribution of configurations such that at least one large square has no holes to the trace in Eq. (3) is at most ϵ. Thus, for such an l_β we can ignore such configurations and restrict to a sum over valid configurations giving an approximation to ρ with error at most ϵ in trace norm, describing ρ as an incoherent sum over trivial states; if desired, one can write this sum in the form of Eq. (2).

Operational properties.—Consider first $T = 0$. The four dimensional toric code on a torus has a degenerate ground state. There are surface operators U_x and U_z that act like the Pauli operators σ_x and σ_z on a two-dimensional subspace of the ground state space. These operators anticommute so that $\{U_x, U_z\} = 0$. Further, defining U_x' and U_z' to denote the same operators translated a distance $L/2$ perpendicular to the given surfaces, the ground state expectation value of $U_x' U_x^\dagger$ equals 1, and similarly for $U_z' U_z^\dagger$. At nonzero temperature, the possibility of correcting errors [13] implies that we can "thicken"[21] those surfaces, giving unitary operators V_x and V_z which are supported within some distance (say, $L/8$) of the given surface which have similar properties at $T > 0$ to the operator U_x and U_z at $T = 0$. In particular, we expect that for sufficiently small T

$$\mathrm{tr}\,(\rho V_x V_z V_x^\dagger V_z^\dagger) \approx -1, \tag{9}$$

$$\mathrm{tr}\,(\rho V_x' V_x^\dagger) \approx \mathrm{tr}(\rho V_z' V_z^\dagger) \approx 1. \tag{10}$$

However, Eqs. (9) and (10), are inconsistent with having an (R, ϵ) trivial state for R sufficiently small compared to L and ϵ of order unity: if ρ is an incoherent mixture of states $\psi_{\mathrm{triv}}(a)$, $\mathrm{tr}(\rho V_x' V_x^\dagger) = \sum_a P(a) \langle \psi_{\mathrm{triv}}(a) | V_x' V_x^\dagger | \psi_{\mathrm{triv}}(a) \rangle$. Then, since the separation between the supports of V_x' and $V_x^\dagger$ is greater than R, this equals $\sum_a P(a) \times \langle \psi_{\mathrm{triv}}(a) | V_x' | \psi_{\mathrm{triv}}(a) \rangle \langle \psi_{\mathrm{triv}}(a) | V_x^\dagger | \psi_{\mathrm{triv}}(a) \rangle$.

For this sum to be close to 1 as in Eq. (10), a ψ_{triv} chosen at random must be, with probability close to 1, an approximate eigenstate of both V_x and V_z. However, this is inconsistent with Eq. (9). One can derive a similar inconsistency result for certain defect creation processes similar to those in [22].

Conversely, we can comment on this usefulness of a trivial state for storing quantum information. To manipulate quantum information at $T > 0$, we need operators V_x and V_y which act like the Pauli matrices on a single qubit, as in Eq. (9). However, since Eqs. (9) and (10), are inconsistent with having a trivial state, we cannot have also corresponding operators V_x', V_y'. Further, such a trivial mixed state can be no more useful for storing information than a trivial pure state. Note that it seems too much to hope for the converse statement (that a nontrivial state is useful for storing quantum information), since even at $T = 0$ the Chern insulator [23] provides an example of a circuit nontrivial state that has a unique ground state and so cannot be a quantum memory.

Discussion.—We have proposed a definition of topological order $T > 0$. While this definition is simple, it allows us to make precise statements about how quantum information can be manipulated in two-dimensional Hamiltonians which are sums of commuting terms. This raises the question of what can happen for arbitrary two-dimensional Hamiltonians. In three dimensions, we expect that discrete gauge theories are topologically trivial at $T > 0$ under the circuit definition (see Supplemental Material [24]). A much more interesting question is whether Haah's code [25,26], which avoids this gauge theory paradigm, is trivial or not at $T > 0$. Using quasiadiabatic continuation [27], we can relate the circuit definition at $T = 0$ to whether or not one can deform one local Hamiltonian into another while avoiding a phase transition; however, an analogous question for $T > 0$ is open (whether the absence of a phase transition in thermodynamic quantities when lowering the temperature from infinity to some finite T_f implies that the density matrix at T_f is trivial). Finally, we ask if there is a circuit definition for exotic critical points [28].

I thank D. Poulin for useful comments.

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