Abatement of Thermal Noise due to Internal Damping in 2D Oscillators with Rapidly Rotating Test Masses

R. Pegna,[1] A. M. Nobili,[2,1] M. Shao,[3] S. G. Turyshev,[3] G. Catastini,[4] A. Anselmi,[4]
R. Spero,[3] S. Doravari,[5] G. L. Comandi,[1] and A. De Michele[2]
[1]*INFN, Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy*
[2]*Department of Physics "E. Fermi", University of Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy*
[3]*Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr., Pasadena, California 91109, USA*
[4]*Thales Alenia Space Italia, Strada Antica di Collegno 253, 10146 Torino, Italy*
[5]*California Institute of Technology, Pasadena, California 91109, USA*
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Mechanical oscillators can be sensitive to very small forces. Low frequency effects are up-converted to higher frequency by rotating the oscillator. We show that for 2-dimensional oscillators rotating at frequency much higher than the signal the thermal noise force due to internal losses and competing with it is abated as the square root of the rotation frequency. We also show that rotation at frequency much higher than the natural one is possible if the oscillator has 2 degrees of freedom, and describe how this property applies also to torsion balances. In addition, in the 2D oscillator the signal is up-converted above resonance without being attenuated as in the 1D case, thus relaxing requirements on the read out. This work indicates that proof masses weakly coupled in 2D and rapidly rotating can play a major role in very small force physics experiments.

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Physics experiments for the measurement of small forces are ultimately limited by thermal noise due to internal losses in the mechanical suspensions (see [1], Sec. IV). Once all systematics are reduced below the signal—and if read out noise is not a limitation—it sets the length of the integration time required for the signal to emerge above thermal noise. A factor 10 better sensitivity—i.e. a 10 times smaller force to be detected—requires an integration time 100 times longer, which makes reduction of thermal noise a must if extremely weak forces are to be detected.

Consider a 2D harmonic oscillator made of two point-like test bodies of reduced mass μ coupled by a spring of stiffness k in both directions of the plane. The general solution is an elliptic orbit with the center in the common center of mass of the bodies, which can be decomposed into the sum of two simple harmonic motions with $\omega_n = \sqrt{k/\mu}$ the frequency of *natural* (or *proper*) oscillations of the test masses relative to each other in each direction.

The oscillator is designed to be sensitive to very small forces acting between the masses in their plane of motion. Therefore, it has a very low natural frequency ω_n (because the sensitivity improves as ω_n^{-2}) and employs springs of very high mechanical quality (i.e., their losses are very small). Moreover, it is operated in vacuum at low residual pressure in order to reduce damping resulting from Brownian motion and with sufficient magnetic shielding to reduce damping from eddy currents in moving conductors ([1], Sec. IV). Such a system is dominated by internal damping.

According to Nyquist fluctuation-dissipation theorem, in the frequency domain the power spectral density (PSD) of the thermal noise force is given (using the "hat" symbol for the Fourier transform) by

$$\langle|\hat{F}_{\mathrm{th}}(\omega)|^2\rangle = 4K_B T\gamma(\omega) \tag{1}$$

with K_B the Boltzmann constant, T the thermal equilibrium temperature and $\gamma(\omega)$ the damping coefficient which, for systems dominated by internal damping has been found to be frequency dependent and given by (1)

$$\gamma(\omega) \simeq \frac{k\phi(\omega)}{\omega}, \tag{2}$$

where ϕ is known as *loss angle* (its modulus is the inverse of the mechanical quality factor Q) which also depends on the frequency ω, albeit mildly, and $\phi(\omega)$ is an odd function of ω. (2) is verified experimentally (see, e.g., [2,3]) and the divergence at zero frequency is a known issue of no relevance in real systems ([1], Sec. VII).

Let ω_{signal} be the frequency of the very small force to be sensed by the oscillator, typically smaller than its natural frequency ($\omega_{\mathrm{signal}} < \omega_n$). Once the experiment is limited by thermal noise due to internal damping, because of the frequency dependence (2), from (1) the *relevant* thermal noise random force (i.e., its component acting on the test masses at the same frequency as the signal) after an integration time t_{int} is

$$\mathfrak{F}_{\mathrm{th}}(\omega_{\mathrm{signal}})|_{t_{\mathrm{int}}} \simeq \sqrt{\frac{4K_B T\mu\omega_n^2\phi(\omega_{\mathrm{signal}})}{\omega_{\mathrm{signal}}}}\frac{1}{\sqrt{t_{\mathrm{int}}}}, \tag{3}$$

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showing that the lower is the frequency of the signal, the longer is the integration time required to bring thermal noise below the signal.

The difficulties of detecting low frequency effects can be mitigated by up-converting the signal to higher frequency. This is achieved by rotating the mechanical oscillator at a frequency faster than that of the signal. Let us therefore consider a 2D harmonic oscillator, with test bodies of equal mass m for simplicity, rotating around an axis perpendicular to its a, b sensitive plane with angular velocity $\omega_{\rm spin}$ with respect to the inertial frame whose x, y plane coincides with the sensitive plane of the oscillator (Fig. 1). The signal is at frequency $\omega_{\rm signal}$ in the inertial frame and it is $\omega_{\rm signal} \ll \omega_{\rm spin}$.

For the oscillator of Fig. 1 we study the effect on the relative motion of the test masses of the force due to thermal noise when the system is in thermal equilibrium at temperature T, with the purpose of assessing its relevance at the frequency of the signal.

We express the motion of the system, subject to the mechanical thermal noise force of the rotating springs, in the inertial x, y reference frame in the frequency domain and in matrix form as follows:

$$\mathbf{D}(\omega)\hat{\vec{r}} = \mathcal{F}(R(\omega_{\rm spin}t)\vec{F}_{\rm th}(t))(\omega), \tag{4}$$

where $\mathbf{D}(\omega)$ is the dynamical matrix of the equations of motion of the system, $\mathcal{F}$ is the Fourier transform operator, $\vec{F}_{\rm th}(t)$ is the thermal noise force due to losses in the rotating springs, and $R(\omega_{\rm spin}t)$ is the 2 by 2 rotation matrix of angle $\omega_{\rm spin}t$:

$$\begin{aligned} R(\omega_{\rm spin}t) &= \begin{pmatrix} \cos(\omega_{\rm spin}t) & -\sin(\omega_{\rm spin}t) \\ \sin(\omega_{\rm spin}t) & \cos(\omega_{\rm spin}t) \end{pmatrix} \\ &= \frac{1}{2}e^{i\omega_{\rm spin}t}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{1}{2}e^{-i\omega_{\rm spin}t}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}. \end{aligned} \tag{5}$$

By defining

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{6}$$

we can write

$$\begin{aligned} \mathbf{D}(\omega)\hat{\vec{r}} &= \mathcal{F}(R(\omega_{\rm spin}t)\vec{F}_{\rm th}(t)) \\ &= \mathbf{A}\hat{\vec{F}}_{\rm th}(\omega + \omega_{\rm spin}) + \mathbf{A}^*\hat{\vec{F}}_{\rm th}(\omega - \omega_{\rm spin}), \end{aligned} \tag{7}$$

where superscript * denotes the complex conjugate. We can see that the effect produced on the dynamical system $\mathbf{D}$ (in the inertial x, y frame) by the rotating thermal noise force $\vec{F}_{\rm th}$ is a linear combination of $\hat{\vec{F}}_{\rm th}(\omega + \omega_{\rm spin})$ and $\hat{\vec{F}}_{\rm th}(\omega - \omega_{\rm spin})$. The most straightforward way to evaluate the components of the thermal noise force in the inertial frame is to write the time average of the cross spectral density (CSD) matrix. Then—in the reasonable assumption of statistical independence of the different vectorial and frequency components of the thermal noise force—we get

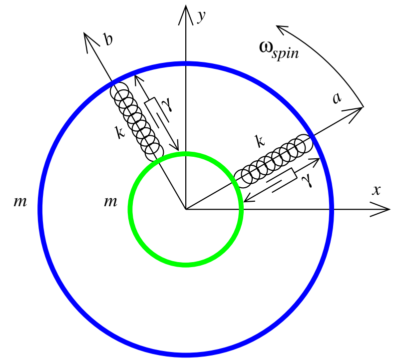

FIG. 1 (color online). Sketch of the 2D rotating oscillator for which thermal noise is evaluated. The proof masses are concentric and rotate—together with the springs—at angular velocity $\omega_{\rm spin}$. They are assumed for the moment as perfectly centered on the rotation axis. The springs are modeled as ideal springs of elastic constant k; to each spring is associated a corotating thermal noise force generator $F_{\rm th}$ and an ideal noiseless damper γ. x, y is the inertial frame; a, b is the rotating one.

$$\begin{aligned} \langle \hat{\vec{F}}_{\rm th}(\omega)\hat{\vec{F}}_{\rm th}(\omega)^\dagger \rangle &= \frac{1}{2}\frac{4K_BTk\phi(\omega + \omega_{\rm spin})}{(\omega + \omega_{\rm spin})}\mathbf{A} \\ &\quad + \frac{1}{2}\frac{4K_BTk\phi(\omega - \omega_{\rm spin})}{(\omega - \omega_{\rm spin})}\mathbf{A}^*, \end{aligned} \tag{8}$$

where $\hat{\vec{F}}_{\rm th}(\omega)^\dagger$ denotes the transpose conjugate of $\hat{\vec{F}}_{\rm th}(\omega)$.

Let us now consider the signal force of interest $\vec{F}_{\rm signal}(t)$ acting on the test masses relative to each other at a very low frequency $\omega_{\rm signal} \ll \omega_{\rm spin}$ in the inertial frame:

$$\vec{F}_{\rm signal}(t) = F_{\rm signal}(\cos(\omega_{\rm signal}t), \sin(\omega_{\rm signal}t)). \tag{9}$$

In the frequency domain (using Dirac δ symbol) it reads

$$\hat{\vec{F}}_{\rm signal}(\omega) = \frac{1}{2}F_{\rm signal}\begin{pmatrix} \delta(\omega - \omega_{\rm signal}) + \delta(\omega + \omega_{\rm signal}) \\ -i\delta(\omega - \omega_{\rm signal}) + i\delta(\omega + \omega_{\rm signal}) \end{pmatrix}. \tag{10}$$

The CSD matrix of the signal is then

$$\begin{aligned} \langle \hat{\vec{F}}_{\rm signal}(\omega)\hat{\vec{F}}_{\rm signal}(\omega)^\dagger \rangle &= \tfrac{1}{2}F_{\rm signal}^2[\delta(\omega - \omega_{\rm signal})\mathbf{A} \\ &\quad + \delta(\omega + \omega_{\rm signal})\mathbf{A}^*]. \end{aligned} \tag{11}$$

By comparing (11) with (8) we can see that only the components of noise at the frequency of the signal, i.e., those with $\omega = \omega_{\rm signal}$ and $\omega = -\omega_{\rm signal}$ do compete with it. By evaluating the diagonal matrix elements of (8) at the signal frequencies we obtain the PSD of the x, y components of the noise competing with the corresponding components of the signal (11). That is, we must compare

$$\frac{1}{2}\left[\frac{4K_BTk\phi(\pm\omega_{\rm signal}+\omega_{\rm spin})}{(\pm\omega_{\rm signal}+\omega_{\rm spin})} + \frac{4K_BTk\phi(\pm\omega_{\rm signal}-\omega_{\rm spin})}{(\pm\omega_{\rm signal}-\omega_{\rm spin})}\right] \quad \text{with} \quad \frac{1}{2}F^2_{\rm signal}. \quad (12)$$

Since we are in the condition $\omega_{\rm signal} \ll \omega_{\rm spin}$, it is apparent that in (12) the dependence on $\omega_{\rm signal}$ disappears and only that on $\omega_{\rm spin}$ remains; moreover, the off diagonal elements of the CSD (8) are very small. In these conditions the x, y components of the thermal noise force are almost uncorrelated and by averaging the x with the y component of the signal we gain a factor $\sqrt{2}$ in the signal-to-noise ratio. Thus, the spectral density of the thermal noise force competing with the signal is

$$\langle|\hat{F}_{\rm th}|\rangle \simeq \sqrt{\frac{4K_BTk\phi(\omega_{\rm spin})}{\omega_{\rm spin}}} \quad (13)$$

and the actual thermal force after an integration $t_{\rm int}$ is

$$\mathfrak{F}'_{\rm th}(\omega_{\rm signal} \ll \omega_{\rm spin})|_{t_{\rm int}} \simeq \sqrt{\frac{4K_BT\mu\omega_n^2\phi(\omega_{\rm spin})}{\omega_{\rm spin}}}\frac{1}{\sqrt{t_{\rm int}}} \quad (14)$$

the major advantage with respect to (3) being that the frequency of the signal is now replaced by the much higher rotation frequency of the oscillator; in addition, losses at higher frequency are found to be smaller than at lower frequency.

Though we have taken great care in a rigorous derivation of this result there is nothing mysterious about it: the energy of thermal noise is the same as at zero spin—simply, its component at the frequency of the signal is much smaller than at zero spin due to the frequency dependence (2) of internal damping.

An example of thermal noise reduction by rotation comes from torsion balances used to test the Equivalence Principle by detecting the twist angle produced by tiny differential forces acting in the horizontal plane. Quite remarkably, they have been able to reach the level of thermal noise ([4], Fig. 20), finding that thermal noise competing with the signal obeys (13) at the rotation frequency of the balance (which is about $2/3$ of its natural torsion frequency, and is the frequency at which the signal is shifted to) and that it has the same $1/\sqrt{\omega}$ dependence at lower frequencies, at which thermal noise dominates (at higher frequencies read out noise dominates instead). Their ([4], eq. 57) computed at the rotation frequency of the balance is the same as our (13); obviously, they measure a thermal noise torque, not force, and k is a torsion constant. By rotating the balance with a period of about 20 min they have improved by a factor 70 as compared to relying on the 24-h rotation of the Earth, reducing the integration time by the same factor. Traditional attempts at reducing thermal noise from internal losses have involved cooling down the apparatus in order to reduce the thermal equilibrium temperature T. However, cryogenics can reduce the integration time by a factor 100 at most, while rotation can do much better than that, and rotating torsion balances have already achieved almost that much.

So far we have referred to a 2D rotating oscillator in which the proof masses are perfectly centered on the rotation axis. In reality perfect centering is impossible; we represent such manufacturing imperfections by an offset vector $\vec{\epsilon}$ of the reduced mass μ from the rotation axis ($\vec{\epsilon}$ is fixed in the rotating frame). At equilibrium the position vector reads

$$\vec{r}_{\rm eq} = \frac{1}{1-(\omega_{\rm spin}/\omega_n)^2}\vec{\epsilon}, \quad (15)$$

which for rotation at frequency much higher than the natural one becomes

$$\vec{r}_{\rm eq} \simeq -\vec{\epsilon}\left(\frac{\omega_n}{\omega_{\rm spin}}\right)^2, \quad (16)$$

showing that the center of mass of the rotating body reaches equilibrium much closer to the rotation axis than it was by construction, by the factor $(\omega_n/\omega_{\rm spin})^2 \ll 1$. This autocentering property is what makes fast rotation more advantageous than the slow one. However, the minus sign indicates that for the equilibrium position to be reached the center of mass of the body must be allowed to move in the rotating plane till it sets itself antiparallel to $\vec{\epsilon}$, as required by (16): if constrained along a single direction it will not autocenter and be strongly unstable, as it has been known since a long time ([5], Ch. 6).

Let us now write and solve the equations of motion of the 2D rotating oscillator around the equilibrium position in the presence of a force, like the signal, of very low frequency. In the inertial frame they read

$$\mu\ddot{\vec{r}} + \gamma_{\omega_{\rm spin}}(\dot{\vec{r}} - \vec{\omega}_{\rm spin}\times\vec{r}) + k\vec{r} = \vec{F}, \quad (17)$$

where $\gamma_{\omega_{\rm spin}}$ is the small internal damping (2) of the oscillator rotating at $\omega_{\rm spin}$; $\vec{F}$ is the signal force whose frequency is so small compared to both $\omega_{\rm spin}$ and ω_n that we assume a constant force for simplicity. In the two-body oscillator of Fig. 1, if the bodies have equal mass m the reduced mass is $m/2$, the natural frequency is $\omega_n = \sqrt{k/(m/2)}$ with the external force acting between them. In the assumptions made ($\omega_{\rm spin} \gg \omega_n$ and very small internal losses) the solution of the homogeneous part of (17) is

$$\vec{r}_w(t) \simeq A_0 e^{\phi_{\omega_{\rm spin}}\omega_n t/2}\begin{pmatrix}\cos(\omega_n t+\varphi_A)\\ \sin(\omega_n t+\varphi_A)\end{pmatrix} + B_0 e^{-\phi_{\omega_{\rm spin}}\omega_n t/2}\begin{pmatrix}\cos(-\omega_n t+\varphi_B)\\ \sin(-\omega_n t+\varphi_B)\end{pmatrix} \quad (18)$$

(with amplitudes and phases determined by initial conditions), showing that in the inertial reference frame the oscillator performs a combination of a forward and a backward orbital motion—known as *whirl motion*—at the (slow) natural frequency ω_n, and the radii of such orbits are exponentially decaying in the case of the backward whirl and exponentially growing in the case of the forward one. We have written the exponential behavior in terms of the small loss angle:

$$\phi_{\omega_{\rm spin}} \simeq \frac{\gamma_{\omega_{\rm spin}}\omega_{\rm spin}}{\mu\omega_n^2} = \frac{\gamma_{\omega_{\rm spin}}\omega_{\rm spin}}{k}. \tag{19}$$

The forward whirl is then a very weak instability. Every natural (or whirl) period the radius of the forward whirl grows by the fraction $\pi\phi_{\omega_{\rm spin}}$, hence the tangential force which produces the growth is—in modulus—$\phi_{\omega_{\rm spin}}kr$, which is a very small fraction of the elastic force, requiring a correspondingly small force to stabilize it. Its frequency is the natural one and does not interfere with the signal (see [6,7]).

In the presence of an external constant force $\vec{F}$, the equations of motion (17) show that (in the inertial frame) the body is displaced to the position

$$\vec{r}_F(t) = \frac{1}{1+\frac{\gamma^2_{\omega_{\rm spin}}\omega^2_{\rm spin}}{k^2}}\left(\frac{\vec{F}}{k} - \frac{\gamma_{\omega_{\rm spin}}}{k^2}\vec{\omega}_{\rm spin}\times\vec{F}\right)$$
$$\simeq \frac{\vec{F}}{k} - \phi_{\omega_{\rm spin}}\frac{\vec{\omega}_{\rm spin}}{\omega_{\rm spin}}\times\frac{\vec{F}}{k}. \tag{20}$$

As we can see, the applied force $\vec{F}$ gives rise to a displacement $\vec{F}/k$ (i.e., the displacement is inversely proportional to the natural frequency squared) and unaffected by rotation, with an additional effect in the orthogonal direction due to rotation which is negligible because of the very small loss angle $\phi_{\omega_{\rm spin}}$. In the rotating frame of the oscillator this constant displacement observed in the inertial one appears at the rotation frequency $\omega_{\rm spin} \gg \omega_n$, yet it is apparent that no attenuation occurs. Instead, it is well known that for an oscillator with 1 degree of freedom, the displacement due to a force at frequency $\omega_{\rm spin} \gg \omega_n$ drops off as $(\omega_n/\omega_{\rm spin})^2$. Note that the signal-to-thermal noise ratio is the same in the two cases, since the displacement due to the signal and that due to the thermal noise force are either both unchanged (by the 2D oscillator) or both attenuated (by the 1D oscillator). When dealing with extremely weak effects a signal whose strength is not attenuated by rotation has the advantage to loosen the requirements on the performance of the read out, as long as rapid rotation takes care of reducing thermal noise.

The general solution of the 2D rotating oscillator in the inertial frame—including the autocentered position (16) fixed on the rotating oscillator itself—is

$$\vec{r}(t) \simeq -\vec{\epsilon}(\omega_{\rm spin}t)\left(\frac{\omega_n}{\omega_{\rm spin}}\right)^2 + \frac{\vec{F}}{k} - \phi_{\omega_{\rm spin}}\frac{\vec{\omega}_{\rm spin}}{\omega_{\rm spin}}\times\frac{\vec{F}}{k}$$
$$+ A_0 e^{\phi_{\omega_{\rm spin}}\omega_n t/2}\begin{pmatrix}\cos(\omega_n t+\varphi_A)\\ \sin(\omega_n t+\varphi_A)\end{pmatrix}$$
$$+ B_0 e^{-\phi_{\omega_{\rm spin}}\omega_n t/2}\begin{pmatrix}\cos(-\omega_n t+\varphi_B)\\ \sin(-\omega_n t+\varphi_B)\end{pmatrix}, \tag{21}$$

which is helpful to comment as follows. Assume zero losses and no external force: only the first term is not zero and the solution is the autocentered position rotating at frequency $\omega_{\rm spin}$; if the force signal $\vec{F}$ is added—still with zero losses—the term $\vec{F}/k$ is not zero and the oscillator is displaced by this vector with autocentering holding as before; finally, if small losses occur—after the backward whirl has died out, and neglecting the small effect $\propto \phi_{\omega_{\rm spin}}$—the forward whirl slowly grows around the displaced position at frequency ω_n. By controlling this weak instability, rotation (and signal modulation) at a frequency much higher than the natural one are achieved with no signal attenuation and thermal noise reduction according to (13).

These findings indicate that mechanical oscillators with concentric proof masses weakly coupled in 2D and rapidly rotating can play a major role in physics experiments for the measurement of extremely weak forces. There is no question that having 2 degrees of freedom—as sketched in Fig. 1—instead of being constrained in 1 direction (while rotating perpendicular to it), is the key dynamical feature of the oscillator which makes fast rotation physically possible, thus ensuring up-conversion of the signal to much higher frequency where the competing thermal noise due to internal losses is much smaller.

In this 2D vs 1D analysis, torsion balances are a special case. As a torque sensor the balance has 1 degree of freedom, hence any torque applied above its (low) torsion resonance frequency is attenuated. However, as a pendulum it has 2 degrees of freedom, with an oscillation period of few seconds. By spinning the pendulum above its oscillation frequency—being allowed to move in the plane—it will self center on the rotation axis minimizing disturbances due to centrifugal forces; this equilibrium position will be stable, save for the weak whirl instability which can be controlled. Torques due to imperfections of the balance in rotation must still be taken care of, but this is an interesting physical property of the torsion balance—in addition to its low torsion frequency and nearly perfect rejection of common mode forces—which has gone unnoticed so far. Although the rotation frequency required to achieve self centering is much higher than the current one, by a few orders of magnitude, if a clever solution is found to improve the read out enough to overcome signal attenuation, this possibility is worth investigating as a very effective alternative to cryogenics, considering the reduction of signal-to-thermal noise ratio.

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[1] P. R. Saulson, Phys. Rev. D **42**, 2437 (1990).
[2] K. Numata, A. Kemery, and J. Camp, Phys. Rev. Lett. **93**, 250602 (2004).
[3] G. Cagnoli *et al.*, Phys. Lett. A **255**, 230 (1999).
[4] E. G. Adelberger *et al.*, Prog. Part. Nucl. Phys. **62**, 102 (2009).
[5] J. P. Den Hartog, *Mechanical Vibrations* (Dover Publications, Inc., New York 1985, first published in 1934).
[6] S. H. Crandall, J. Sound Vib. **11**, 3 (1970).
[7] A. M. Nobili *et al.*, Classical Quantum Gravity **16**, 1463 (1999).