A Positive-Energy Theorem for Einstein-Aether and Hořava Gravity

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Energy positivity is established for a class of solutions to Einstein-aether theory and the IR limit of Hořava gravity within a certain range of coupling parameters. The class consists of solutions where the aether 4-vector is divergence-free on a spacelike surface to which it is orthogonal (which implies that the surface is maximal). In particular, this result holds for spherically symmetric solutions at a moment of time symmetry.

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INTRODUCTION.—It is difficult to modify general relativity (GR) in a fashion that meets basic theoretical requirements such as stability, energy positivity, and the existence of a well-posed initial value formulation. Even in the linearized theory one or more of these properties often fails. Analyzing them in the full, nonlinear theory is of course much more difficult, so much so that it is rarely done.

In this Letter, we shall establish a fully nonlinear positive-energy result for two closely related modifications of GR whose properties and predictions have been extensively studied over the past several years, "Einstein-aether theory" and Hořava gravity. The first theory, called "ae-theory" for short, consists of a dynamical unit timelike vector "aether" field u^a coupled to Einstein gravity [1,2] (for reviews see [3,4]). The vector can be thought of as the 4-velocity of a preferred frame; it spontaneously breaks local Lorentz symmetry since, being a unit vector, it is everywhere nonzero in any solution, including flat spacetime. Hořava gravity [5] (for a review see [6]) can be viewed as general relativity coupled to a preferred time function T, restricted by invariance under reparametrizations of T. Thus, it depends on T only via the unit normal (timelike) vector field $NT_{,a}$, with $N = (g^{mn}T_{,m}T_{,n})^{-1/2}$. Here we consider the so-called "nonprojectable" version of that theory, in which the lapse function N is allowed to be an arbitrary function of position, and we include in the action all terms consistent with the symmetry of foliation preserving diffeomorphisms (so that the theory is dynamically well behaved [7]).

For both theories, we restrict our analysis to terms in the Lagrangian with no more than two derivatives of the metric or the vector field. Hořava gravity theory is then equivalent to a version of ae-theory in which the aether is constrained to be hypersurface orthogonal at the level of the action [7,8]. (Every hypersurface-orthogonal solution of ae-theory is a solution of Hořava gravity. The converse is not true in general, but it does hold in spherical symmetry for solutions with a regular center [9].) Moreover, the total energy of asymptotically flat solutions of the two theories is given by the same expressions in terms of the metric and aether fields (for ae-theory see [10,11], for Hořava gravity see [12,13]), as are the stress tensors [8]. A positive-energy result for ae-theory therefore implies a similar result for Hořava gravity. We thus focus the discussion on ae-theory.

The Lagrangian of ae-theory depends on four dimensionless coupling constants $c_{1,2,3,4}$. In the hypersurface-orthogonal sector of the theory, only the combinations $c_{14} = c_1 + c_4$, $c_{13} = c_1 + c_3$, and c_2 enter. The coupling constants of Hořava gravity can be expressed in terms of these combinations [7,8]. Hyperbolicity, stability, and energy positivity of the linearized theory hold for certain ranges of the coupling constants in ae-theory [10,14–16] and Hořava gravity [7,9]. These ranges coincide in the two theories for the spin-2 and spin-0 modes. (Ae-theory has an additional spin-1 mode.) Here we establish a positive-energy result for the full, nonlinear theory.

One might approach this problem by considering the aether field as simply one more matter field and try to use the usual results for the positivity of mass in general relativity. However, the aether Lagrangian involves the covariant derivative (as opposed to the exterior derivative, which is all that is needed for minimally coupled scalar fields or for electromagnetism). This leads to very different behavior of the action when the metric is varied and, in particular, to the violation of the dominant energy condition for the energy-momentum tensor. Since the dominant energy condition is what is needed for both the Schoen-Yau [17,18] and Witten [19,20] proofs of the positive-energy theorem, these results do not apply to ae-theory, and one might therefore expect that energy is not positive in ae-theory. On the other hand, a spherically symmetric static vacuum solution is known explicitly [21] which has positive energy despite having everywhere negative aether energy density, which suggests that there may be a general positive-energy property.

The question we are addressing here is whether the total energy $M_{æ}$ of asymptotically flat solutions in ae-theory is positive. (It turns out that this energy is not the same as the ADM mass $M_{\rm ADM}$ that defines the total energy in general relativity, although the ratio $M_{æ}/M_{\rm ADM}$ may be a universal constant.) As we will show, this is indeed the case for solutions where the vector field is hypersurface orthogonal and where one of those hypersurfaces is asymptotically flat and has vanishing trace of the extrinsic curvature (i.e., is "maximal"). In particular, this result holds for spherically symmetric solutions at a moment of time symmetry. The possibility of generalizing this result is briefly discussed at the end of the Letter.

The method of proof will be to exploit the result of Schoen and Yau [17], which shows that the ADM mass of an asymptotically flat spatial metric on an orientable three manifold is non-negative if the Ricci scalar is non-negative. Although the Ricci scalar of the physical 3-metric is *not* generally non-negative, we will find a conformally related 3-metric with a positive Ricci scalar and whose ADM mass is equal to $M_{æ}$ of the original spacetime.

EINSTEIN-AETHER THEORY.—The action for Einstein-aether theory is the most general generally covariant functional of the spacetime metric g_{ab} and aether field u^a involving no more than two derivatives (not including total derivatives),

$$S = \int \sqrt{-g}(L_{æ} + L_m)d^4x, \tag{1}$$

where

$$L_{æ} = \frac{1}{16\pi G}[R - K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n + \lambda(g_{ab}u^a u^b + 1)] \tag{2}$$

and L_m denotes the matter Lagrangian. Here R is the Ricci scalar, $K^{ab}{}_{mn}$ is defined as

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m - c_4 u^a u^b g_{mn}, \tag{3}$$

where the c_i are dimensionless coupling constants, and λ is a Lagrange multiplier enforcing the unit timelike constraint on the aether. The convention used in this Letter for the metric signature is $(- + ++)$ and the units are chosen so that the speed of light defined by the metric g_{ab} is unity.

The field equations from varying (1) with respect to g^{ab}, u^a, and λ are given, respectively, by

$$G_{ab} = T^{æ}_{ab} + 8\pi G T^m_{ab}, \tag{4}$$

$$\nabla_a J^a{}_b + \lambda u_b + c_4 a_a \nabla_b u^a = 0, \tag{5}$$

$$u^a u_a = -1. \tag{6}$$

Here G_{ab} is the Einstein tensor of the metric g_{ab} and T^m_{ab} is the matter stress tensor. The quantities $J^a{}_b$, a_a and the aether stress energy $T^{æ}_{ab}$ are given by

$$J^a{}_m = K^{ab}{}_{mn}\nabla_b u^n, \tag{7}$$

$$a_a = u^b \nabla_b u_a, \tag{8}$$

$$\begin{aligned} T^{æ}_{ab} = {} & \lambda u_a u_b + c_4 a_a a_b - \tfrac{1}{2} g_{ab} J^c{}_d \nabla_c u^d \\ & + c_1(\nabla_a u_c \nabla_b u^c - \nabla^c u_a \nabla_c u_b) \\ & + \nabla_c[J^c{}_{(a} u_{b)} + u^c J_{(ab)} - J_{(a}{}^c u_{b)}]. \end{aligned} \tag{9}$$

In the weak-field, slow-motion limit, ae-theory reduces to Newtonian gravity with a value of Newton's constant $G_{\rm N}$ related to the parameter G in the action (1) by $G_{\rm N} = G(1 - c_{14}/2)^{-1}$ [22]. Note that a sensible Newtonian limit requires that $c_{14} < 2$.

The total energy of an asymptotically flat solution, defined in the asymptotic aether rest frame, is given by

$$M_{æ} = M_{\rm ADM} - \frac{c_{14}}{8\pi G}\int_\infty r^a a_a, \tag{10}$$

where $M_{\rm ADM}$ is the usual ADM mass (16), the integral is over a two-sphere at infinity, and r^a is a unit vector in the radial direction. (The total energy was first found by Eling [10] using pseudotensor methods, and then by Foster [11] using Wald's Noether charge method [23,24]. It is written in the above form in [11].) At least in the weak-field, slow-motion limit, we have $G_{\rm N} M_{æ} = G M_{\rm ADM}$. That is, the difference between $M_{æ}$ and $M_{\rm ADM}$ is accounted for by the difference between G_N and G. We suspect that the equality $G_{\rm N} M_{æ} = G M_{\rm ADM}$ holds in general (i.e., not just in the weak-field slow-motion limit) and therefore that the positivity of $M_{æ}$ is equivalent to the positivity of $M_{\rm ADM}$ when $c_{14} < 2$. However, for the purposes of this Letter we will only address the question of the positivity of $M_{æ}$.

HYPERSURFACE-ORTHOGONAL CASE.—We consider here only solutions where u^a is hypersurface orthogonal. This is always the case in spherical symmetry, but more generally it is a bona fide restriction. On an asymptotically flat slice orthogonal to u^a, the spatial metric h_{ab} and extrinsic curvature K_{ab} are given by

$$h_{ab} = g_{ab} + u_a u_b, \tag{11}$$

$$K_{ab} = -h_a{}^c \nabla_c u_b. \tag{12}$$

The trace of the extrinsic curvature is given by

$$K = -\nabla_a u^a. \tag{13}$$

If u^a is orthogonal to surfaces of constant t for some function t, then $u_a = N\nabla_a t$ for some "lapse" function N. Crucial for our purposes here is the fact that in this case the acceleration vector $a_a = u^b \nabla_b u_a$ is equal to a spatial gradient,

$$a_a = D_a \ln N, \tag{14}$$

where D_a is the spatial derivative operator.

Using this expression for the acceleration of the aether, the total energy (10) becomes

$$M_{æ} = M_{\rm ADM} - \frac{c_{14}}{8\pi G}\int_\infty r^i \partial_i N. \tag{15}$$

The energy takes this simple form because it is defined and expressed in the asymptotic aether rest frame, and we have chosen the t coordinate so that $N \to 1$ at infinity. Note that the second term in this expression for the aether mass is similar to the change of the ADM mass under a conformal transformation. Consider a conformally transformed metric $\tilde{h}_{ab} = \Omega^2 h_{ab}$ where $\Omega \to 1$ at infinity. Since the ADM mass is given by

$$M_{\rm ADM} = \frac{1}{16\pi G}\int_\infty r^i(\partial_j h_{ji} - \partial_i h_{jj}), \tag{16}$$

it follows that under a conformal transformation we have

$$\tilde{M}_{\rm ADM} = M_{\rm ADM} - \frac{1}{4\pi G}\int_\infty r^i \partial_i \Omega. \tag{17}$$

Therefore, $M_{æ}$ is equal to the ADM mass of a conformally transformed metric,

$$M_{æ} = \tilde{M}_{\rm ADM}, \qquad \tilde{h}_{ab} = N^{c_{14}} h_{ab}, \tag{18}$$

using the conformal factor $\Omega = N^{c_{14}/2}$. The question of whether $M_{æ}$ is positive thus becomes that of whether $\tilde{M}_{\rm ADM}$ is positive.

As in general relativity, the uu component of the Einstein equation (4) turns out to be an initial value constraint equation in the present setting where u^a is orthogonal to the spatial surface. (This is not *a priori* obvious, since the aether stress tensor (9) contains second time derivatives. The spherical case was treated in detail in [25] and a general argument is given in [26].) This equation reads

$$^{(3)}R + K^2 - K^{ab}K_{ab} = 2(T^{æ}_{ab} + 8\pi G T^m_{ab})u^a u^b, \tag{19}$$

where $^{(3)}R$ is the scalar curvature of the spatial metric. Using the fact that u^a is orthogonal to the surface, the uu component of the aether stress tensor (9) may be evaluated as

$$2T^{æ}_{ab}u^a u^b = c_{14}(2D_a a^a + a_a a^a) - c_2 K^2 - c_{13}K_{ab}K^{ab}, \tag{20}$$

where D_a is the covariant derivative with respect to the spatial metric. On substituting Eq. (20) into Eq. (19) one finds

$$\begin{aligned} ^{(3)}R = {} & 16\pi G\rho + c_{14}(2D_a a^a + a_a a^a) \\ & + (1 - c_{13})K_{ab}K^{ab} - (1 + c_2)K^2, \end{aligned} \tag{21}$$

where $\rho = T^m_{ab}u^a u^b$ is the matter energy density.

POSITIVE-ENERGY THEOREM.—Now if $\rho \geq 0$ and $K = 0$, then in ordinary general relativity ($c_{1,2,3,4} = 0$) this implies $^{(3)}R \geq 0$, so the theorem of Schoen and Yau (SY) [17] implies that the ADM energy is positive. In Einstein-aether theory, provided c_{14} and $1 - c_{13}$ are positive, the $a_a a^a$ and $K_{ab}K^{ab}$ terms contribute positively, but the term $D_a a^a$ has indefinite sign. Thus, we cannot expect a definite sign for the ADM mass. However, recall that it is the aether mass $M_{æ}$ (15) that is the physical mass of the spacetime, and $M_{æ}$ is equal to the ADM mass of a conformally transformed metric (18).

Remarkably, precisely the same conformal transformation that yields $M_{æ} = \tilde{M}_{\rm ADM}$ removes the indefinite term $D_a a^a$ of (21). To see this, note that the Ricci scalar of $\tilde{h}_{ab} = \Omega^2 h_{ab}$ is related to $^{(3)}R$ by

$$^{(3)}\tilde{R} = \Omega^{-2}[^{(3)}R - 4D^a D_a \ln\Omega - 2(D^a \ln\Omega)(D_a \ln\Omega)]. \tag{22}$$

With the conformal factor $\Omega = N^{c_{14}/2}$ we have $D_a \ln\Omega = (c_{14}/2)a_a$, so (21) and (22) together yield

$$\begin{aligned} ^{(3)}\tilde{R} = {} & N^{-c_{14}}[16\pi G\rho + c_{14}(1 - c_{14}/2)a_a a^a \\ & + (1 - c_{13})K_{ab}K^{ab} - (1 + c_2)K^2]. \end{aligned} \tag{23}$$

The result of SY thus implies that the ADM energy of $\tilde{h}_{ab}$ is positive, and therefore the aether mass $M_{æ}$ of h_{ab} is positive, provided $\rho \geq 0$, $K = 0$, $0 \leq c_{14} \leq 2$, and $c_{13} \leq 1$.

These inequalities on $c_{1,2,3,4}$ are required by the stability and positive energy of the linearized theory. What we have found here is that they also suffice to imply the positive energy of hypersurface-orthogonal configurations on maximal slices of the fully nonlinear theory.

The SY theorem holds when the spatial manifold has any number of asymptotically flat "ends." This provides a way to extend the result to the case when the spatial metric at a moment of time symmetry has a minimal surface with a singularity inside. One can just smoothly join a second copy of the space to itself along the minimal surface, thus obtaining a space, without the original interior of the minimal surface, to which the theorem applies for each end. Thus, the mass of the spherical static vacuum solution, which possesses a minimal 2-sphere with a singularity inside, must be positive, as indeed it was found to be by explicit construction [21]. This is an instructive example, since the aether energy density is negative everywhere in the solution.

We now consider how the above result can be generalized. One such generalization is to remove the condition that $K = 0$. For general relativity this was done by Schoen and Yau [18] using a technique that essentially reduced the problem to one covered by their first proof (but required the dominant energy condition, which is stronger than the condition $\rho > 0$). We expect that the method of [18] can also be used for ae-theory and therefore that the condition that the slice be maximal can be removed. If so, the theorem could be extended to cover, in particular, time dependent spherical solutions with R^3 topology, and possibly spherical black holes [27,28].

More generally, one might hope to remove the condition that the aether vector field is hypersurface orthogonal (this condition always holds in Hořava gravity). Since the positive mass theorem is essentially a property of the constraint equations, to find a general positive mass theorem one would have to examine the general constraint equations in ae-theory. These equations were first written in [29] using a result from [30] (see also [26] for a different derivation). They are complicated, so it might be better to start with a simple subcase, such as that of a moment of time symmetry, to see whether a positive-energy result could be obtained there.

Finally, it is worth emphasizing that a key step in our proof of energy positivity was to express the total energy for ae-theory in terms of the ADM mass of a particular conformally related spatial metric whose Ricci scalar is positive under the conditions of the theorem. It is an unexpected fact that, in the hypersurface-orthogonal case, the same conformal transformation that makes the energy $M_{æ}$ equal to the ADM mass $\tilde{M}_{\mathrm{ADM}}$ removes the indefinite sign divergence term in the Ricci scalar. This may be a hint that a similar conformal transformation could be used to generalize the result.

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