Stable Pfaffian State in Bilayer Graphene

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Here, we show that the incompressible Pfaffian state originally proposed for the $\frac{5}{2}$ fractional quantum Hall states in conventional two-dimensional electron systems can actually be found in a bilayer graphene at one of the Landau levels. The properties and stability of the Pfaffian state at this special Landau level strongly depend on the magnetic field strength. The graphene system shows a transition from the incompressible to a compressible state with increasing magnetic field. At a finite magnetic field of ~10 T, the Pfaffian state in bilayer graphene becomes more stable than its counterpart in conventional electron systems.

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Ever since the discovery of the quantum Hall state at the Landau level (LL) filling factor $\nu = \frac{5}{2}$, the first even-denominator state observed in a single-layer system, it has been very aptly characterized as an ''enigma'' [1]. It was clear at the outset that this state must be different from the fractional quantum Hall effect (FQHE) in predominantly odd-denominator filling fractions [2,3]. Understanding this enigmatic state has remained a major challenge in all these years [4]. At this half-filled first excited LL [5], a novel state described by a pair wave function involving a Pfaffian [6,7] (or anti-Pfaffian [8]) has been the strongest candidate, as supported by extensive numerical analysis [9]. More intriguing are the elementary charged excitations at this ground state that have a charge $e^* = e/4$ and obey ''non-Abelian'' statistics [10–12]. Recent observation of the $e^* = e/4$ quasiparticle charge at the $\nu = \frac{5}{2}$ quantum Hall state [13] has brought the issue to the fore [14]. Experimental observation of the corresponding neutral modes [15] supports their non-Abelian statistics. It has been suggested that these non-Abelian quasiparticles, besides carrying the signatures of Majorana fermions [16] in this system, might even be useful for quantum information storage and processing in an intrinsically fault-tolerant manner [17].

Electrons in recently discovered graphene [18] display a range of truly remarkable behavior [19]. The dynamics of electrons in monolayer graphene, a hexagonal honeycombed lattice of carbon atoms, is that of massless Dirac fermions with linear dispersion, chiral eigenstates, valley degeneracy, and unusual LLs in an external magnetic field [19]. Theoretical studies of FQHE in monolayer [20] and bilayer graphene [21] were reported earlier by us. Experimental observations of the $\nu = \frac{1}{3}$ FQHE in monolayer graphene [22,23] have provided a glimpse of the role highly correlated electrons play in graphene. Given the acute interest in studying the properties of the $\nu = \frac{5}{2}$ state in conventional two-dimensional electron gas (2DEG), a natural question to ask is how does this state manifest itself in graphene? Below, we show that the $\frac{5}{2}$-Pfaffian state is, in fact, more stable in graphene than in conventional two-dimensional electron systems.

The Pfaffian state, which was proposed as an incompressible ground state of a half-filled LL, is written as

$$\Psi_{\mathrm{Pf}} = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2 \exp\left(-\sum_i \frac{z_i^2}{4\ell_0^2}\right),$$

where the Pfaffian is defined as [6,7]

$$\mathrm{Pf}\, M_{ij} = \frac{1}{2^{N/2}(N/2)!}\sum_{\sigma\in S_N}\mathrm{sgn}\sigma\prod_{l=1}^{N/2} M_{\sigma(2l-1)\sigma(2l)}$$

for an $N \times N$ antisymmetric matrix whose elements are M_{ij}; S_N is the group of permutations of N objects, $\ell_0 = (\hbar/eB)^{(1/2)}$ is the magnetic length, and $z = x - iy$. The $\nu = \frac{1}{2}$ Pfaffian ground state is an exact ground state with zero energy for a special three-particle interaction which is nonzero only if all three particles are in close proximity to each other [7]. For a two-particle interaction, it is important to find a realistic interaction potential whose many-particle ground state is well-described by the Pfaffian state.

For the conventional (*nonrelativistic*) 2DEG, the incompressible state, related to the Pfaffian states, has been observed only at a half-filled second LL, i.e., at the total filling factor $\nu = \frac{5}{2}$. To determine the relation of the $\nu = \frac{5}{2}$ ground state with the Pfaffian state, the 2DEG at $\nu = \frac{5}{2}$ has been extensively studied numerically for a finite number of electrons [14]. A relatively good (but not 100%) overlap with the Pfaffian state was found. The overlap can be improved by varying the interelectron potential, for example, by increasing the thickness of the two-dimensional layer [24]. By varying the interaction function, the close

proximity to the Pfaffian function with an overlap of 99% can be achieved, but that is obtainable only for unrealistic interaction potential shapes. We show below that, in bilayer graphene, the ground state at one of the LLs is well-described by the Pfaffian state with an overlap of more than 92%, which makes bilayer graphene the desired system to observe the $\frac{5}{2}$-Pfaffian state.

The interaction properties of a two-dimensional system are commonly determined by the Haldane pseudopotentials [25], which are the energies, V_m, of two electrons with relative angular momentum m. The $\nu = \frac{1}{2}$ Pfaffian state is most sensitive to the lowest pseudopotentials, V_1, V_3, and V_5. The Haldane pseudopotentials make it convenient to study the finite-size systems numerically in the spherical geometry [3]. The size of the sphere and the number of single-particle states are determined by the parameter S, where $2S$ is the number of magnetic fluxes through the sphere in units of the flux quanta. The single-electron states are characterized by the angular momentum S and its z component S_z. For a many-electron system, the corresponding states are classified by the total angular momentum L and its z component [26]. For a system with N electrons, the $\nu = \frac{1}{2}$ Pfaffian state is realized at $2S = 2N - 3$. Here, the filling factor $\nu = \frac{1}{2}$ is defined as the filling factor of a given LL. In the spherical geometry, the $\nu = \frac{1}{2}$ Pfaffian state is the exact ground state for a three-particle interaction potential that is nonzero only if the total angular momentum of three particles is $3S - 3$. It is described by the interaction Hamiltonian $H_{\rm int} = \frac{e^2}{\kappa \ell_0} \sum_{i<j<k} P_{ijk}(3S-3)$, where $P_{ijk}(L)$ is the three-particle projection operator onto the state with total angular momentum L [7].

For a single graphene layer, the LL wave functions are mixtures of those for LLs of nonrelativistic systems; for example, the first LL in graphene can be expressed in terms of the zeroth and the first LL wave functions of the nonrelativistic system [20]. Our numerical analysis of finite-size systems in a spherical geometry with up to 14 electrons shows that the largest excitation gap of about $0.02e^2/(\kappa \ell_0)$ occurs at the $n = 2$ graphene LL. Although the excitation gap at a finite-size system in this case is comparable to the $\nu = \frac{5}{2}$ nonrelativistic system, the overlap of the ground state with the Pfaffian state is less than 0.5 at all LLs. This means that an incompressible $\nu = \frac{1}{2}$ Pfaffian state is unlikely to be found in monolayer graphene.

We show here that, in bilayer graphene, on the other hand, the stability of the $\nu = \frac{1}{2}$ Pfaffian state is greatly enhanced. Notably, one of the bilayer LLs (for a given valley) has a stable $\nu = \frac{1}{2}$ Pfaffian state, whose properties can be controlled by a magnetic field. The maximum overlap of the finite system ground state with the corresponding Pfaffian state occurs at finite values of the magnetic field. We found that the $\nu = \frac{1}{2}$ incompressible state of a bilayer graphene is more stable than the corresponding state in a conventional two-dimensional system.

We consider a bilayer graphene which consists of two coupled graphene layers with the Bernal stacking arrangement. Each graphene layer has two sublattices, say, A and B. For the Bernal stacking arrangement, the coupling is mainly between the atoms of sublattice A of the lower layer and atoms of sublattice B' of the upper layer. The coupling is determined by the interlayer hopping integral, t. For one projection of spin, e.g., $+\frac{1}{2}$, the state of the bilayer graphene can be expressed in terms of the four-component spinor $(\psi_A, \psi_B, \psi_{B'}, \psi_{A'})^T$ for valley K and $(\psi_{B'}, \psi_{A'}, \psi_A, \psi_B)^T$ for valley K'. The subindices A, B and A', B' correspond to lower and upper layers, respectively. The properties of bilayer graphene can be controlled by a bias voltage, ΔU, which is the potential difference between the upper and lower layers [19].

The discrete eigenstates of the bilayer Hamiltonian are labeled by the index $n = 0, 1, 2, \ldots$ [27,28]. For a given value of n, there are four bilayer LLs. The corresponding wave functions can be expressed in terms of n, $|n-1|$, and $n+1$ conventional LL wave functions [27,28]. We present our results for $N = 8$, 10, and 14 electron systems. To determine the incompressibility of the system, we calculated the excitation gap and the overlap of the ground state wave function with the Pfaffian function. We consider only one valley, for example, valley K. The results are similar for K'.

For all but one bilayer LL, the overlap of the $\nu = \frac{1}{2}$ ground state with the Pfaffian state is found to be small (< 0.6). At the same time, there is one special LL (for each valley) at which the $\nu = \frac{1}{2}$ ground state is well-described by the Pfaffian function. This level corresponds to one of the LLs with $n = 0$. At a small bias voltage, ΔU, the wave function corresponding to this LL is the mixture of conventional LL wave functions with indices 0 and 1. It has the form $(\phi_0, 0, 0, (t/\sqrt{2})\phi_1)$, where ϕ_n are nth nonrelativistic LL functions and t is in units of $\hbar v_F/\ell_0$. At small values of the dimensionless hopping integral, $t(\ell_0/\hbar v_F)$, the interaction within this level is similar to the one at the lowest LL of a conventional system, which does not show an incompressible $\nu = \frac{1}{2}$ state. At large values of $t(\ell_0/\hbar v_F)$, the special bilayer LL is similar to the $n = 1$ LL of the conventional system and shows the $\nu = \frac{1}{2}$ Pfaffian state. By varying the magnetic field, the dimensionless interlayer hopping integral is changed, which modifies the interaction within the LL and changes the properties of the $\nu = \frac{1}{2}$ state. We present our results only for this special LL.

At the zero bias voltage, this special LL has zero energy and is degenerate with another level, which has the form $(0, 0, 0, \phi_0)$. In addition to this accidental degeneracy, each level has twofold valley degeneracy, which makes the zero energy state fourfold degenerate. At a finite bias voltage, this degeneracy is completely lifted and the special LL of the bilayer can be isolated. In Fig. 1(a), we show

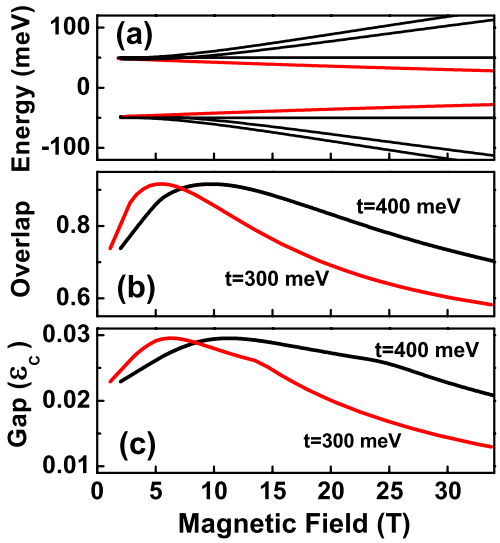

FIG. 1 (color online). (a) Few lowest LLs of a bilayer graphene, shown for $\Delta U = 100$ meV and $t = 400$ meV. The two solid red (light gray) lines belonging to different valleys correspond to the LLs where the $\nu = \frac{1}{2}$ Pfaffian state can be observed. (b) Overlap of the exact many-particle ground state with the Pfaffian function. (c) Excitation gap of the $\nu = \frac{1}{2}$ state. The results are for $N = 14$ and $2S = 25$ and the zero bias voltage. The black and red (light gray) lines correspond to $t = 400$ meV and 300 meV, respectively. Here, the energy unit is $\varepsilon_c = e^2/\kappa\ell_0$.

the lowest LLs of a bilayer at a finite bias voltage. Two solid red (light gray) lines correspond to the special LLs of the two valleys. The many-particle properties of these two levels are identical.

In Figs. 1(b) and 1(c), we show the magnetic field dependence of the overlap of the $\nu = \frac{1}{2}$ ground state with the Pfaffian state and the corresponding excitation gap. At a small magnetic field, the dimensionless hopping integral is large and the system becomes similar to the conventional system at the $n = 1$ LL. With an increasing magnetic field, the properties of the system change nonmonotonically and the overlap with the Pfaffian state reaches its maximum at a magnetic field of $\sim$10 T (and for $t = 400$ meV). The overlap at this point is ≈ 0.92, which is a major improvement over the nonrelativistic system (~ 0.75). The dimensionless hopping integral at this point is $t(\ell_0/\hbar v_F) \approx 4.89$.

At a large magnetic field, the system is close to the $n = 0$ nonrelativistic LL, the overlap with the Pfaffian state becomes small, and the $\nu = \frac{1}{2}$ state is finally compressible. This dependence on the magnetic field opens up interesting possibilities to investigate the stability and appearance and disappearance of the $\nu = \frac{1}{2}$ Pfaffian state in a single bilayer LL. Although the Pfaffian state becomes unstable only at large magnetic fields, this property strongly depends on the value of the hopping integral. At smaller hopping integrals, the magnetic field range of stability of the Pfaffian state shrinks. For example, at $t = 300$ meV, the Pfaffian state is expected to be unstable

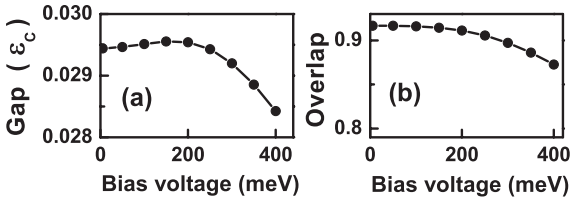

FIG. 2. (a) The excitation gap (b) and the overlap with the Pfaffian state versus the bias voltage, ΔU. Here, $N = 14$ and $2S = 25$, and the magnetic field is 10 T.

at $B \sim 40$ T (see Fig. 1). Another parameter which controls the properties of the graphene bilayer is the bias voltage. Although the bias voltage modifies the bilayer wave functions, we found that the overlap of the ground state with the Pfaffian state and the excitation gap have weak dependence on the bias voltage within a broad range of ΔU (see Fig. 2). The overlap monotonically decreases with increasing ΔU, which suppresses the overlap by only a few percent. The large excitation gap and the large overlap observed for different system sizes are shown in Fig. 3.

The stability and the strength of the Pfaffian state can be also analyzed in terms of the general dependence of the pseudopotentials, V_m, on the relative angular momentum, m. We characterize the interaction potential of the bilayer graphene by two parameters: V_1/V_5 and V_3/V_5 [14]. These parameters depend on the magnetic field. By varying the magnetic field, this dependence can be shown as a line in the $(V_1/V_5) - (V_3/V_5)$ plane (Fig. 4). That line connects the initial point at $B = 0$ to the final point, corresponding to a large magnetic field, $B = \infty$. The $\nu = \frac{1}{2}$ bilayer graphene system at the initial and final points are identical to the conventional systems at the first ($n = 1$) and zero ($n = 0$) LLs, respectively. In Ref. [14], the region of the compressible $\nu = \frac{1}{2}$ state and the region of strong overlap with the Pfaffian state were identified (see Fig. 4). With an increasing magnetic field, the $\nu = \frac{1}{2}$ bilayer graphene system transforms from a $\nu = \frac{5}{2}$ nonrelativistic state (at small values of B) to a more stable incompressible state with large overlap, and finally to a compressible state (at a large magnetic field). For the hopping integral $t = 400$ meV, the transition from the incompressible to a compressible $\nu = \frac{1}{2}$ state occurs at $B \sim 100$ T.

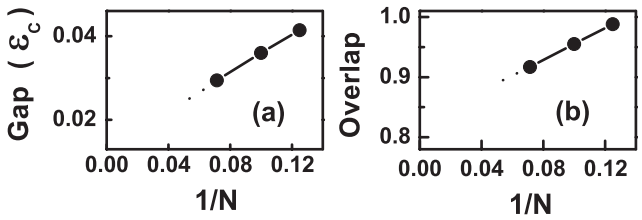

FIG. 3. (a) The excitation gap and (b) the overlap with the Pfaffian state are shown for different number of electrons: $N = 8$, 10, and 14. The magnetic field is 10 T, and the bias voltage is zero.

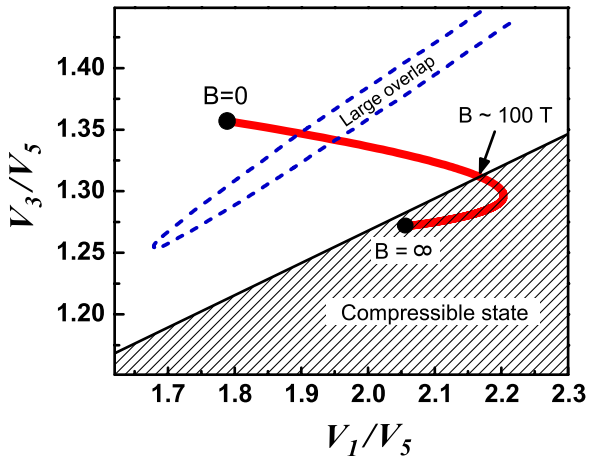

FIG. 4 (color online). Trajectory of the interelectron interaction with varying magnetic fields, shown by a thick solid red (light gray) line in the plane $(V_1/V_5) - (V_3/V_5)$ for the "special" LL of the bilayer graphene. The initial point of the trajectory (at $B = 0$) corresponds to the nonrelativistic system at $n = 1$ LL, while the final point (at $B = \infty$) corresponds to the nonrelativistic system at the $n = 0$ LL. The shaded region illustrates the compressible $\nu = \frac{1}{2}$ state, while the blank region corresponds to the incompressible $\nu = \frac{1}{2}$ state (Ref. [14]). The crossing of the boundary between the compressible and incompressible states occurs at $B \sim 100$ T for the hopping integral $t = 400$ meV. The blue (dark gray) dashed line shows the region of large overlap with the Pfaffian state (Ref. [14]).

In the above analysis, we have considered the properties of the electron system in a single LL, taking the projection of interaction potential on a given LL. In this approach, we disregard the interaction-induced mixing of LLs. The strength of this mixing is determined by the parameter $\lambda = (e^2/\kappa\ell_0)/\Delta_n$, where Δ_n is the inter-LL energy separation. In bilayer graphene, this parameter is approximately inversely proportional to $B^{1/2}$ and decreases with increasing magnetic field. For $\kappa \approx 4$, λ is close to unity at $B \approx 12$ T. Although, at smaller magnetic fields, $B < 12$ T and $\lambda > 1$, from the analysis of LL mixing in conventional systems [29,30], it is safe to conclude that, for up to $\lambda = 3$, the changes in the overlap with the Pfaffian state are less than 20%. We expect that our results should be qualitatively correct even at small magnetic fields, e.g., $B > 2$ T.

In conclusion, a stable incompressible $\nu = \frac{1}{2}$ Pfaffian state can in fact be observed in a bilayer graphene only at one LL. The properties of this state strongly depend on the value of the magnetic field. With an increasing magnetic field, the $\nu = \frac{1}{2}$ state transforms from an incompressible state at a small magnetic field to a compressible state at a large magnetic field. At intermediate values of the magnetic field, $B \sim 10$ T, the $\nu = \frac{1}{2}$ state becomes much more stable than the corresponding state in a conventional two-dimensional electron system. Therefore, bilayer graphene is an unique physical system to study the equally unique, incompressible $\nu = \frac{1}{2}$ Pfaffian state.

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