Generality of Deterministic Chaos, Exponential Spectra, and Lorentzian Pulses in Magnetically Confined Plasmas

J. E. Maggs and G. J. Morales
Physics and Astronomy Department, University of California, Los Angeles, Los Angeles, California 90095, USA
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The dynamics of transport at the edge of magnetized plasmas is deterministic chaos. The connection is made by a previous survey [M. A. Pedrosa *et al.*, Phys. Rev. Lett. **82**, 3621 (1999)] of measurements of fluctuations that is shown to exhibit power spectra with exponential frequency dependence over a broad range, which is the signature of deterministic chaos. The exponential character arises from Lorentzian pulses. The results suggest that the generalization to complex times used in studies of deterministic chaos is a representation of Lorentzian pulses emerging from the chaotic dynamics.

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Although it is not widely appreciated by the plasma science community, since the early 1980s it has been recognized by researchers in several disciplines [1–3] that an intrinsic and observable signature of systems whose dynamics exhibits deterministic chaos [4] is a fluctuation power spectrum with an exponential frequency dependence, i.e., $P(\omega) \propto \exp(-2\omega\tau)$, where τ is a time constant associated with the underlying processes. The temporal signals associated with these spectra are intermittent or "spiky," consisting of a series of apparently randomly occurring "spikes" or pulses. Deterministic chaos is a nonlinear dynamical state that arises when the amplitude of a few collective coherent modes is sufficiently large to induce chaotic trajectories in the associated phase space. Exponential spectra have been identified in widely different systems including the fluctuation in sunspot number [5], CO_2 chaotic forcing of ice ages [6], the unipolar injection hydrodynamic instability [7], turbulence in neurons related to Parkinson's disease [8], weakly turbulent Couette-Taylor flows [9,10], and Rayleigh-Bénard convection [11], among others. In magnetized plasmas chaotic dynamics can arise when unstable drift waves driven by the pressure gradients exceed a threshold value [12,13]. The potential fields of the drift waves result in $E \times B$ plasma flows that are necessarily perpendicular to the confining magnetic field. The phase space of such a system is two dimensional.

The association of exponential spectra with deterministic chaos has been firmly established through detailed experiments and numerical solutions of a wide class of nonlinear models [9–11,14], but, surprisingly, at the present time, there is no rigorous mathematical proof that provides a direct link between these two features. It has been identified by researchers [1,15] who have examined the mathematical structure of this challenging problem that the proof requires the analytic continuation of the underlying equations to the complex-time domain. The purpose of the generalization is to extract a singularity that lies near the real axis that allows the evaluation of the Fourier transform. The separation between the pole and the real axis is believed to determine the value of the parameter τ. The procedure works technically, but it fails to provide a connection to an underlying physical process. This Letter emphasizes that recent insight into this issue has emerged from transport experiments in magnetized plasmas, both in a basic linear device [16,17] and in a stellarator, toroidal configuration [18]. In these completely different experiments, fluctuations in the plasma pressure are observed to follow an exponential frequency dependence for frequencies below the ion cyclotron frequency. More importantly, the origin of the observed exponential behavior in the spectrum is due to the fact that the pulses occurring in the intermittent time signals have a Lorentzian functional form. The temporal shape of an individual pulse is

$$L(t) = \frac{A}{1 + (\frac{t-t_0}{\tau})^2} = \frac{A}{2}\left\{\frac{\tau}{\tau + i(t - t_0)} + \frac{\tau}{\tau - i(t - t_0)}\right\}, \tag{1}$$

where A is the peak amplitude of a pulse centered at time t_0 and having width τ. The second form of the Lorentzian given in Eq. (1) explicitly displays the pair of conjugate poles at $t = t_0 \pm i\tau$ that gives rise to the exponential nature of the power spectrum. The power spectrum of a series of N pulses is

$$P(\omega) \propto \left|\sum_{n=1}^{N} \exp(i\omega t_{0,n} - \omega\tau_n)\right|^2. \tag{2}$$

This spectrum is a sum over the residues of a collection of single poles in the complex-time plane and is exponential if the distribution of pulse widths τ_n is sufficiently narrow.

In the plasma experiments, time signals are typically measured at a fixed spatial location by probes and are a manifestation of the effects of spatially extended structures, generated by deterministic chaos, sweeping past the probes. The underlying chaos is associated with coherent drift waves driven unstable by the pressure gradients. A modeling study [19] has shown that retaining two individual modes is sufficient to result in the generation of Lorentzian pulses when the amplitude of the modes exceeds a threshold value. The threshold corresponds to the $E \times B$ velocity imparted by the modes exceeding the phase velocity of the modes (approximately the diamagnetic drift velocity). The experimental observations [16–18] and the model results [19] also show that the value of the parameter τ is a fraction (1/4 to 1/5) of the wave period of the drift modes.

Since the Lorentzian pulses are typically embedded in the coherent fluctuations or other plasma flows, they can display distortions that may hide their true identity when individually sampled in a time series. However, Lorentzian pulses exhibit a robust contribution to the formation of an exponential spectrum. Only systems that exhibit pulses with a shape that closely approximates a Lorentzian, and that have a relatively narrow distribution of pulse widths, can result in an exponential spectrum. To provide a better appreciation for this important property, an example of various pulse shapes is shown in Fig. 1. A general pulse shape can be generated from the inverse Fourier transform of the function [20]

$$\ln(\tilde{\mathcal{L}}(\omega)) = -(\tau|\omega|)^{\alpha}[1 + isF(\omega, \alpha, \tau)] + i\omega t_0, \quad \text{where}$$
$$F(\omega, \alpha, \tau) = \begin{cases} \mathrm{sgn}(\omega)\tan(\pi\alpha/2)[(\tau|\omega|)^{1-\alpha} - 1] & \alpha \neq 1 \\ -2/\pi \ln(|\tau\omega|) & \alpha = 1, \end{cases} \tag{3}$$

with $\mathrm{sgn}(\omega) = -1$, $\omega < 0$, $\mathrm{sgn}(\omega) = 0$, $\omega = 0$, and $\mathrm{sgn}(\omega) = 1$, $\omega > 0$. The function $\tilde{\mathcal{L}}(\omega)$ contains four parameters that characterize the pulse: α, the shape parameter ($0 < \alpha \leq 2$), s, the skewness ($-1 \leq s \leq 1$), τ, the width ($0 \leq \tau \leq \infty$), and t_0, the displacement ($-\infty \leq t_0 \leq \infty$). The shape parameter α allows for a continuous variation in pulse shape from a Gaussian ($\alpha = 2$) to a

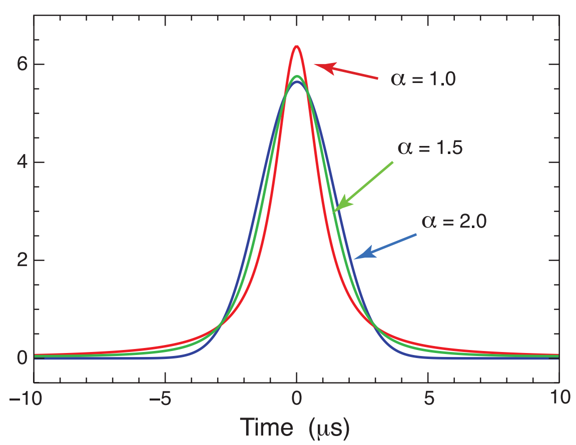

FIG. 1 (color). Examples of pulse shapes for three values of the parameter α in Eq. (3). Gaussian is $\alpha = 2.0$, Lorentzian is $\alpha = 1.0$, and intermediate shape $\alpha = 1.5$. The "tails" of the pulses become more prominent as α decreases from 2.0 to 1.0. All pulses have the same width, $\tau = 1.0$ μs.

Lorentzian ($\alpha = 1$). The corresponding frequency power spectra associated with the pulses are shown in Fig. 2. Figure 2(a) displays the results in the popular double-logarithmic format used in turbulence studies motivated by Kolmogorov's influential work [21] that predicts a power-law dependence, and Fig. 2(b) displays the same results in a log-linear format that sensitively identifies an exponential spectrum because it is a simple straight line. The choice of the specific time scale in Fig. 1 is for comparison to the experimental survey by Pedrosa *et al.* [22] shown later.

It is seen from Fig. 2 that the spectrum of a Lorentzian pulse extends over a larger frequency range than that of a corresponding Gaussian pulse or even an intermediate pulse ($\alpha = 1.5$), as expected. But the double-logarithmic format, because of its large frequency-scale compression, does not exhibit a significantly different qualitative behavior in the power spectra of the three pulse shapes. In fact, in this presentation it is tempting to interpret a purely exponential spectrum ($\alpha = 1$) as being a sequence of power laws with varying indices, as is often concluded in turbulence studies that attempt to interpret the phenomena in terms of scalings motivated by Kolmogorov's work [21] or other turbulence models. For example, in the survey by

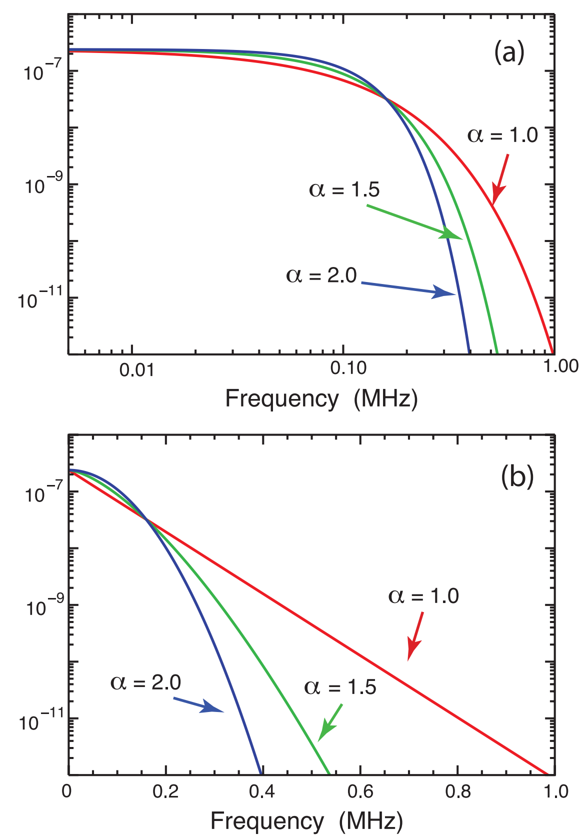

FIG. 2 (color). (a) Power spectra of the pulse shapes in Fig. 1 in a double-logarithmic format. (b) Same spectra in a log-linear format. The exponential spectrum is easily identified as a straight line in the log-linear format.

Pedrosa *et al.* [22] the double-logarithmic format was employed, and spectral indices of -1, and -3 were identified as present in different frequency bands. It was then suggested that multifractal scaling may be needed to explain the results. In contrast, the log-linear display shown in Fig. 2 illustrates the fundamental distinction between the linear shape of the power spectrum of a Lorentzian pulse and the curved shape of the power spectra produced by other pulse shapes. Pulses with $\alpha > 1$ exhibit a characteristic concave curvature (i.e., downward, towards lower values) in the low-frequency domain.

At a fixed value of the shape parameter α, the temporal symmetry of a pulse is determined by the skewness parameter s. Symmetric pulses have zero skewness, $s = 0$. The solid curve in Fig. 3(a) shows an example of a Lorentzian pulse with a skewness value of $s = -0.8$, which results in a pulse shape with a pronounced "leading edge," a feature emphasized by some plasma researchers [23–25]. It is important to note, however, that the skewness of a pulse does not alter the shape of the power spectrum. From Eq. (3) it is clear the skewness parameter only appears as a phase term in the Fourier transform, and thus does not appear in the power spectrum (the square of the absolute value of the Fourier transform). Indeed, skewed pulses are routinely observed in the time signals from plasma probes. Figure 3(b) presents an example of a skewed pulse observed in the experiments of Pace *et al.* [17] (solid curve), fit with a skewed Lorentzian pulse (dashed curve) obtained from the inverse Fourier transform of the expression given in Eq. (3). A very good fit is obtained to the experimentally observed pulse with the parameter values $(\alpha, s, \tau, t_0) = (1.0, -.45, 8.0\ \mu\text{s}, 0.0)$. The Fourier transform of the skewed pulse, however, is a straight line in a log-linear plot.

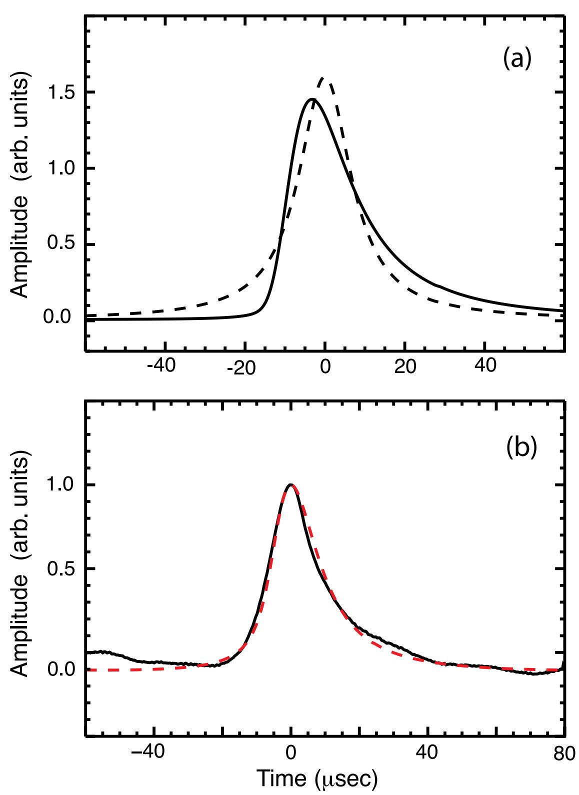

FIG. 3 (color online). (a) Skewed Lorentzian pulse (solid curve) with a pronounced "leading edge" ($s = -0.8$) is compared to a symmetric pulse (dashed curve) with the same width, 8 μs. (b) An experimentally observed pulse (solid line) is compared to a pulse generated from the inverse Fourier transform of $\tilde{\mathcal{L}}(\omega)$ (dashed curve).

To concretely illustrate the connection between deterministic chaos and exponential spectra in a magnetized plasma, Fig. 4 presents the results of a simple, two-mode (azimuthal mode numbers $m = 1$ and $m = 6$) model of the relaxation of a magnetized temperature filament of the type investigated by Pace *et al.* [16,17]. The amplitude of the $m = 1$ mode is increased adiabatically before ramping up the $m = 6$ mode amplitude. The interaction of the two modes leads to chaotic Lagrangian orbits once an amplitude threshold is exceeded. The top panel shows the complex, but spatially connected, structures formed when the $m = 1$ mode is at full amplitude and the $m = 6$ mode amplitude is just below the threshold for chaotic behavior. The region of elevated temperature near the center (orange) corresponds to orbits in the "island of stability" associated with the $m = 1$ mode. The middle panel shows the fine-scale spatial structures that develop after the onset of chaos. The bottom panel is the frequency spectrum of the temperature fluctuations at a time corresponding to the middle panel, showing a clear exponential dependence in a log-linear display, as highlighted by the red dashes. The protruding peaks correspond to the fundamental and first few harmonics of the coherent modes driving the chaos.

The extensive survey undertaken by Pedrosa *et al.* [22] provides a major, worldwide synthesis of the observed behavior of fluctuations at the edge of magnetically confined plasmas. The survey focuses on toroidal devices that explore fusion physics. The breadth of the devices considered is significant; it includes tokamaks and stellarators whose parameters range in magnetic field strength from 0.67 to 2.6 T and plasma densities from 0.5 to $3 \times 10^{19}\ \text{m}^{-3}$. The study by Pedrosa *et al.* [22] attempted to identify a universal frequency dependence for edge fluctuations. The empirical search sought to identify dependencies having a functional form given by their Eq. (2), i.e., $P(\omega) = P_0 g(\lambda\omega)$, where λ is a constant that is device dependent. By adjusting the value of λ it was demonstrated in Figs. 3(a)–3(c) of Ref. [22] that the power spectra of all the devices exhibited identical behavior, thus indicating that they arise from a universal process. As is typical of such studies, the spectra were displayed in double-logarithmic format, and, although the evidence for universality is quite impressive, it was not possible to deduce the sought-after function $g(\lambda\omega)$. Motivated by the recent

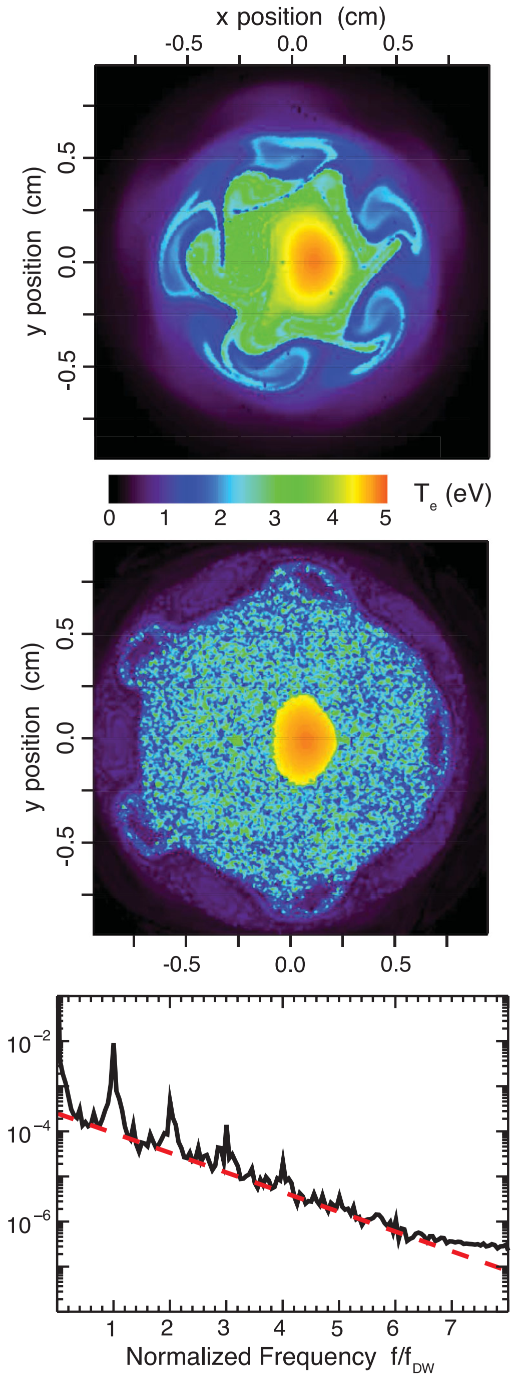

FIG. 4 (color). Results of a deterministic-chaos model. Top: Temperature contours in a filament just before the onset of chaotic behavior. Middle: Contours after the onset of chaos. Bottom: Power spectrum of temperature fluctuations corresponding to the middle panel. f_{DW} is the drift wave frequency.

insight into exponential spectra previously discussed, it is of interest to test if the function g is exponential. Figure 5 provides the desired comparison. The original Figs. 3(a)–3(c) in Ref. [22] are displayed in black. They correspond to the spectra of fluctuations in ion saturation current [5(a)], floating potential [5(b)], and radial turbulent particle flux [5(c)]. The red curve superimposed on the data surveys is essentially the same Lorentzian spectrum shown earlier in Fig. 2(a); it corresponds to a Lorentzian pulse whose width is $\tau = 1\ \mu$s, i.e., the curve labeled $\alpha = 1$ in Fig. 1. The same curve is used in all three panels. Remarkably, it is seen that the Lorentzian spectrum closely matches all three curves in the survey over a significant range of low frequencies. The match with the turbulent flux [5(c)] is nearly perfect. The small deviation in the high-frequency region of 5(a), where the absolute signal is quite small, could be related to the value of the noise-floor level. From this comparison it is evident that the desired universal function that summarizes the well-established universal behavior is an exponential.

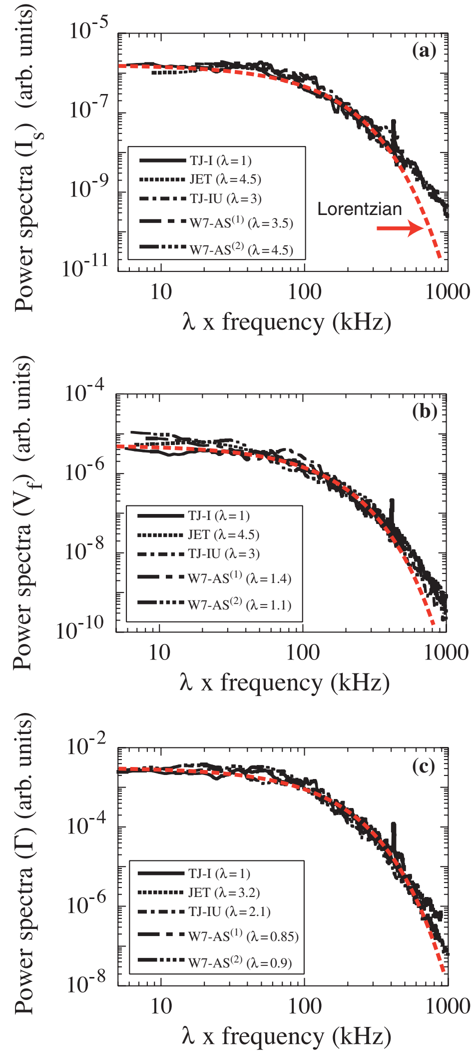

FIG. 5 (color). Power spectra of fluctuations in ion saturation current (a), floating potential (b), and turbulent flux (c) from the survey of Pedrosa *et al.* (black) (used with permission from [22]) is compared to the power spectrum of a single Lorentzian pulse with width, $\tau = 1\ \mu$s (red dotted lines). The power spectra are clearly exponential for frequencies below 400 kHz, and the turbulent flux spectra is exponential over nearly the entire frequency range displayed.

Compelled by the breadth and universality of the data survey of Pedrosa *et al.* [22], and by the independent observation of exponential spectra in controlled studies in a linear device [16,17], it is well warranted to conclude that, in general, the fluctuation spectrum at the edge of magnetically confined plasmas is exponential. Furthermore, because exponential spectra are widely accepted as a signature of deterministic chaos, it is appropriate to deduce that deterministic chaos regulates the underlying dynamics at the edge of magnetically confined plasmas. Turning these results to a broader perspective, because Lorentzian pulses have been identified to be the underlying physical cause of the exponential spectrum, in both linear [16,17] and toroidal geometry [18], this plasma-derived information suggests that the mathematical generalization to complex times used in studies [1,15] of deterministic chaos is a representation of Lorentzian pulses emerging from the chaotic dynamics. Specifically, the poles in the complex-time plane associated with deterministic chaos come in complex conjugate pairs, and power spectra arising from deterministic chaos have the general form given in Eq. (2). Although the connection to Lorentzian pulses has not been made in deterministic-chaos experiments in fluid systems, their presence can be seen in published time signals, such as Fig. 2 of Ref. [9].

In summary, the generality of exponential spectra in magnetized plasmas has been established by extensive and detailed experimental evidence. The underlying connections to deterministic chaos warrant incorporation into contemporary theoretical developments.

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[1] U. Frisch and R. Morf, Phys. Rev. A **23**, 2673 (1981).
[2] H. S. Greenside, G. Ahlers, P. C. Hohenberg, and R. W. Walden, Physica (Amsterdam) **5D**, 322 (1982).
[3] A. Libchaber, S. Fauve, and C. Laroche, Physica (Amsterdam) **7D**, 73 (1983).
[4] H. G. Schuster and W. Just, *Deterministic Chaos* (Wiley-VCH, Weinheim, 2005).
[5] A. Bershadskii, Europhys. Lett. **85**, 49002 (2009).
[6] A. Bershadskii, J. Cosmol. **8**, 1893 (2010) [http://www.JournalofCosmology.com/].
[7] B. Mairaison and P. Atten, Ann. N.Y. Acad. Sci. **404**, 38 (1983).
[8] D. S. Andres, D. F. Cerquetti, and M. Merello, J. Neurosci. Methods **197**, 14 (2011).
[9] A. Brandstater and H. L. Swinney, Phys. Rev. A **35**, 2207 (1987).
[10] C. L. Streett and M. Y. Hussaini, Appl. Numer. Math. **7**, 41 (1991).
[11] M. R. Paul, M. C. Cross, P. F. Fischer, and H. Greenside, Phys. Rev. Lett. **87**, 154501 (2001).
[12] A. M. Batista, I. L. Caldas, S. R. Lopes, R. L. Viana, W. Horton, and P. J. Morrison, Phys. Plasmas **13**, 042510 (2006).
[13] T. Klinger, A. Latten, A. Piel, G. Bonhomme, T. Pierre, and T. Dudok de Wit, Phys. Rev. Lett. **79**, 3913 (1997).
[14] B. Mensour and A. Longtin, Physica (Amsterdam) **113D**, 1 (1998).
[15] D. E. Sigeti, Physica (Amsterdam) **82D**, 136 (1995).
[16] D. C. Pace, M. Shi, J. E. Maggs, G. J. Morales, and T. A. Carter, Phys. Rev. Lett. **101**, 085001 (2008).
[17] D. C. Pace, M. Shi, J. E. Maggs, G. J. Morales, and T. A. Carter, Phys. Plasmas **15**, 122304 (2008).
[18] G. Hornung, B. Nold, J. E. Maggs, G. J. Morales, M. Ramisch, and U. Stroth, Phys. Plasmas **18**, 082303 (2011).
[19] M. Shi, D. C. Pace, G. J. Morales, J. E. Maggs, and T. A. Carter, Phys. Plasmas **16**, 062306 (2009).
[20] Some readers may recognize $\tilde{L}(\omega)$ as the characteristic function of Levy stable probability density distributions.
[21] A. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 16 (1941) [republished as Proc. R. Soc. London **434**, 15 (1991)].
[22] M. A. Pedrosa, C. Hidalgo, B. A. Carreras, R. Balbín, I. García-Cortés, D. Newman, B. van Milligen, E. Sánchez, J. Bleuel, M. Endler, S. Davies, and G. F. Matthews, Phys. Rev. Lett. **82**, 3621 (1999).
[23] G. Y. Antar, G. Counsell, Y. Yu, B. Labombard, and P. Devynck, Phys. Plasmas **10**, 419 (2003).
[24] J. A. Boedo, D. Rudakov, R. Moyer, S. Krasheninnikov, D. Whyte, G. McKee, G. Tynan, M. Schaffer, P. Stangeby, P. West, S. Allen, T. Evans, R. Fonck, E. Hollmann, A. Leonard, A. Mahdavi, G. Porter, M. Tillack, and G. Antar, Phys. Plasmas **8**, 4826 (2001).
[25] D. A. D'Ippolito, J. R. Myra, and S. J. Zweben, Phys. Plasmas **18**, 060501 (2011).