Particle Self-Bunching in the Schwinger Effect in Spacetime-Dependent Electric Fields

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(Received 1 July 2011; published 27 October 2011)

Nonperturbative electron-positron pair creation (the Schwinger effect) is studied based on the Dirac-Heisenberg-Wigner formalism in 1 + 1 dimensions. An *ab initio* calculation of the Schwinger effect in the presence of a simple space- and time-dependent electric field pulse is performed for the first time, allowing for the calculation of the time evolution of observable quantities such as the charge density, the particle number density or the total number of created particles. We predict a new self-bunching effect of charges in phase space due to the spatial and temporal structure of the pulse.

DOI: 10.1103/PhysRevLett.107.180403 PACS numbers: 12.20.Ds, 11.10.Kk, 11.15.Tk

Introduction.—The vacuum of quantum electrodynamics (QED) is unstable against the formation of many-body states in the presence of an external electric field, manifesting itself as the creation of electron-positron pairs [1–3]. This effect has been a long-standing but still unobserved prediction as the generation of near-critical field strengths $E_{\mathrm{cr}} \sim 10^{18}$ V/m has not been feasible so far. Because of the advent of a new generation of high-intensity laser systems such as the European XFEL or the extreme light infrastructure (ELI), this effect might eventually become observable within the next decades.

Previous investigations of the Schwinger effect in the presence of time-dependent electric fields [4–14], space-dependent electric fields [15–19] as well as collinear electric and magnetic fields [20–22] led to a good understanding of the general mechanisms behind the pair creation process by now. However, realistic fields of upcoming high-intensity laser experiments showing both spatial and temporal variations have not been fully considered yet. Only recently it became possible to study the Schwinger effect in such realistic electric fields owing to recent theoretical progress as well as due to the rapid development of computer technology. Specifically, the Dirac-Heisenberg-Wigner phase-space formulation of QED in the presence of an external electric field [23–26] (DHW formalism) has attracted interest again [27–29]. It provides a real-time nonequilibrium formulation of the quantum production process. Also, a one-to-one mapping between the DHW function (phase-space formalism) and the one-particle distribution function (quantum kinetic formalism) exists in the limit of a spatially homogeneous, time-dependent electric field.

The Schwinger effect in the presence of an arbitrary spacetime-dependent electric field is properly described by the DHW formalism in the form of a partial differential equation (PDE) system for the irreducible components of the DHW function. The numerical solution of the PDE system allows for the calculation of any observable quantity in terms of the irreducible components.

In the present work, we consider a simple model for a subattosecond high-intensity laser pulse in standing wave mode with finite extension. In the focus of the beam, pair production along the direction of the electric field gives the dominant contribution to the Schwinger effect. Ignoring particle momenta orthogonal to this dominant direction, the system reduces to a 1 + 1 dimensional setting, which is studied for the first time here and solved numerically [30].

Formalism.—Following the fundamental work of [24], we start with the gauge-invariant equal-time commutator of two Dirac field operators

$$\Phi(x, y, t) := \mathcal{U}(x, y)[\bar{\Psi}(x - y/2, t), \Psi(x + y/2, t)], \quad (1)$$

with x denoting the center-of-mass and y the relative coordinate. Here, the Wilson-line factor which ensures gauge invariance is chosen along a straight line

$$\mathcal{U}(x, y) = \exp\left(-ie \int_{-1/2}^{1/2} d\xi A(x + \xi y, t) y\right). \quad (2)$$

The vector potential $A(x, t)$ is treated as classical mean field; i.e., photon fluctuations are neglected. This approximation is well justified for the pair-production process in QED. Tree-level radiation reactions which might play a sizable role for strong fields according to recent investigations [31–33] are also neglected in this work.

Taking the vacuum expectation value $\langle\Omega|\Phi(x, y, t)|\Omega\rangle$, we trade y for a kinetic momentum variable p by a Fourier transformation. This defines the DHW function

$$\mathcal{W}(x, p, t) := \frac{1}{2} \int dy e^{-ipy} \langle\Omega|\Phi(x, y, t)|\Omega\rangle. \quad (3)$$

Because of the fact that $\mathcal{W}(x, p, t)$ is in the Dirac algebra, it may be decomposed in terms of its Dirac bilinears

$$\mathcal{W}(x, p, t) = \tfrac{1}{2}[\mathbb{S} + i\gamma^5\mathbb{P} + \gamma^\mu\mathbb{V}_\mu], \quad (4)$$

with irreducible components transforming as scalar $\mathbb{S}(x, p, t)$, pseudoscalar $\mathbb{P}(x, p, t)$ and vector $\mathbb{V}_\mu(x, p, t)$. For brevity, these components will later on collectively be denoted as $\mathbb{W}(x, p, t)$. The derivation of the corresponding equations of motion follows that in 3 + 1 dimensions [24,30] and yields the following hyperbolic PDE system

$$\left[\frac{\partial}{\partial t} + \Delta\right]\mathbb{S} - 2p\mathbb{P} = 0, \tag{5}$$

$$\left[\frac{\partial}{\partial t} + \Delta\right]\mathbb{V}_0 + \frac{\partial}{\partial x}\mathbb{V} = 0, \tag{6}$$

$$\left[\frac{\partial}{\partial t} + \Delta\right]\mathbb{V} + \frac{\partial}{\partial x}\mathbb{V}_0 = -2m\mathbb{P}, \tag{7}$$

$$\left[\frac{\partial}{\partial t} + \Delta\right]\mathbb{P} + 2p\mathbb{S} = 2m\mathbb{V}, \tag{8}$$

with the pseudodifferential operator

$$\Delta(x, p, t) = e\int_{-1/2}^{1/2} d\xi E\left(x + i\xi\frac{\partial}{\partial p}, t\right)\frac{\partial}{\partial p}. \tag{9}$$

Along with $\omega(p) = \sqrt{m^2 + p^2}$, the appropriate vacuum initial conditions at asymptotic times $t_{\rm vac} \to -\infty$ are

$$\mathbb{S}_{\rm vac}(p) = -\frac{m}{\omega(p)} \quad \text{and} \quad \mathbb{V}_{\rm vac}(p) = -\frac{p}{\omega(p)}. \tag{10}$$

The irreducible components are not directly observable; however, they constitute the observable quantities which can be derived from Noether's theorem. For our purpose, the charge $\mathcal{Q}(t)$ as well as the energy of the Dirac particles $\mathcal{E}(t)$ are of special interest:

$$\mathcal{Q}(t) = e\int d\Gamma\mathbb{V}_0(x, p, t), \tag{11}$$

$$\mathcal{E}(t) = \int d\Gamma[m\mathbb{S}(x, p, t) + p\mathbb{V}(x, p, t)], \tag{12}$$

with $d\Gamma = dxdp/(2\pi)$ denoting the phase-space volume element. The integrands $q(x, p, t) = \mathbb{V}_0(x, p, t)$ and $\epsilon(x, p, t) = [m\mathbb{S}(x, p, t) + p\mathbb{V}(x, p, t)]$ can be viewed as pseudocharge density and pseudoenergy density, respectively. Because of the fact that we are considering a quantum theory, it is more appropriate to consider the momentum space marginal distributions

$$q(p, t) := \int\frac{dx}{(2\pi)}q(x, p, t), \tag{13}$$

$$\epsilon(p, t) := \int\frac{dx}{(2\pi)}[m\mathbb{S}(x, p, t) + p\mathbb{V}(x, p, t)]. \tag{14}$$

Requiring that the total energy of the Dirac particles should be calculable by integrating a particle number pseudodistribution $n(x, p, t)$ times the one-particle energy $\omega(p)$, it is also useful to introduce the momentum space particle number densities

$$n(p, t) := \int\frac{dx}{(2\pi)}n(x, p, t), \tag{15}$$

with

$$n(x,p,t) = \frac{m[\mathbb{S}(x,p,t) - \mathbb{S}_{\rm vac}(p)] + p[\mathbb{V}(x,p,t) - \mathbb{V}_{\rm vac}(p)]}{\omega(p)}. \tag{16}$$

The vacuum subtractions account for a normalization of the density relative to the vacuum Dirac sea. Accordingly, the total number of created particles reads

$$N(t) = \int dpn(p, t). \tag{17}$$

The PDE system Eqs. (5)–(8) calls for further rewritings or even approximations as arbitrarily high momentum derivatives have to be taken into account in general:

(a) Full solution in conjugate space: As the momentum p appears linearly in the PDE system Eqs. (5)–(8), we can transform these equations to conjugate y space. As a consequence, $\Delta(x, p, t)$ transforms into a function of y as well,

$$\int\frac{dp}{(2\pi)}e^{ipy}\Delta(x, p, t) = -iey\int_{-1/2}^{1/2} d\xi E(x + \xi y, t), \tag{18}$$

resulting in an exact, first order PDE system.

(b) Leading order derivative expansion: The simplest approximation is to expand $\Delta(x, p, t)$ in a series with respect to the spatial variable. Requiring that [27]

$$\left|E(x,t)\frac{\partial\mathbb{W}(x,p,t)]}{\partial p}\right| \gg \frac{1}{24}\left|E''(x,t)\frac{\partial^3\mathbb{W}(x,p,t)]}{\partial p^3}\right|, \tag{19}$$

it is well justified to neglect the higher derivatives:

$$\Delta(x, p, t) \simeq eE(x, t)\frac{\partial}{\partial p}, \tag{20}$$

yielding an approximate, first order PDE system.

(c) Local density approximation: Approximations can also be constructed on the level of the marginal distribution $n(p, t)$. Given an electric field $E(x, t) = E_0 g(x)h(t)$, and assuming that the spatial variation scale is much larger than the Compton wavelength $\lambda \gg \lambda_C$, it is well justified to locally describe the Schwinger effect at any point x independently. We then define the particle number quasidistribution in local density approximation as

$$n_{\rm loc}(x, p, t) := 2\mathcal{F}(p, t; x). \tag{21}$$

$\mathcal{F}(p, t; x)$ denotes the one-particle distribution function which is found by solving the quantum Vlasov equation [34,35] at any fixed point $x_{\rm fixed}$ for a time-dependent electric field $E(t) = E_0 g(x_{\rm fixed})h(t)$. Accordingly,

$$n_{\rm loc}(p, t) := \int\frac{dx}{(2\pi)}n_{\rm loc}(x, p, t). \tag{22}$$

Results.—Our idealized model for a spatially and temporally well-localized laser pulse in a standing wave mode is parameterized by the electric field:

$$E(x, t) = E_0 \exp\left(-\frac{x^2}{2\lambda^2}\right)\mathrm{sech}^2\left(\frac{t}{\tau}\right), \tag{23}$$

with τ and λ denoting the characteristic time and length scale, respectively. We choose the parameters $\tau = 10/m$, $E_0 = 0.5E_{\mathrm{cr}}$ in this investigation, corresponding to an intense subattosecond pulse. As the spatial extent as well as the total energy of the electric field of the pulse decrease with λ, if all other parameters are held fixed, it is convenient to disentangle this trivial scaling effect and investigate scaled quantities for better comparability

$$\bar{n}(p, t) := \frac{n(p, t)}{\lambda} \quad \text{and} \quad \bar{N}(t) := \frac{N(t)}{\lambda}. \tag{24}$$

Full solution vs approximations.—In Fig. 1 we compare the asymptotic value $\bar{n}(p, t \to \infty)$ of the full solution with the leading order derivative expansion as well as with the local density approximation for different values of λ. The difference between the various results is rather small for broader pulses. As the various approximations are in good agreement with the full solution, the pair creation process can indeed be considered as taking place at any point x independently in this regime. For decreasing λ, however, the various results differ substantially.

As expected, the leading order derivative expansion becomes worse for small λ. Whereas a previous study of higher derivative terms signaled a potential failure at large momenta [27], we here observe a breakdown of this approximation for small momenta $p/m \to 0$. For larger λ, the dominant momenta are still well approximated, but for λ approaching λ_C, the truncation artefacts overwhelm the physical values. Also the fact that the particle density $\bar{n}(p, t \to \infty)$ acquires negative values in the derivate expansion signals a clear breakdown of this approximation for small momenta. The local density approximation fails in a different respect: The peak momentum of the full solution is shifted to smaller values for decreasing λ which is not reflected by the local density approximation.

Particle number density.—In Fig. 2, we investigate the behavior of the full solution $\bar{n}(p, t \to \infty)$ for different values of λ. A decreasing λ involves a shift of the peak momentum to a smaller value: The value of the acceleration by the electric field depends on the actual position such that the field excitations feel a varying acceleration when moving through the electric field. Accordingly, the field excitations are less accelerated for narrow pulses.

Moreover, the shape of $\bar{n}(p, t \to \infty)$ becomes higher and narrower for decreasing λ, at least for $\lambda \gtrsim 4\lambda_C$. This is a self-bunching effect caused by the spatial inhomogeneity: Excitations which are created with high momenta are accelerated for a shorter period as they leave the field rapidly. By contrast, excitations which are created with small momenta stay longer inside the field and are accelerated for a longer period. Accordingly, the created particles are bunched into a smaller phase-space volume.

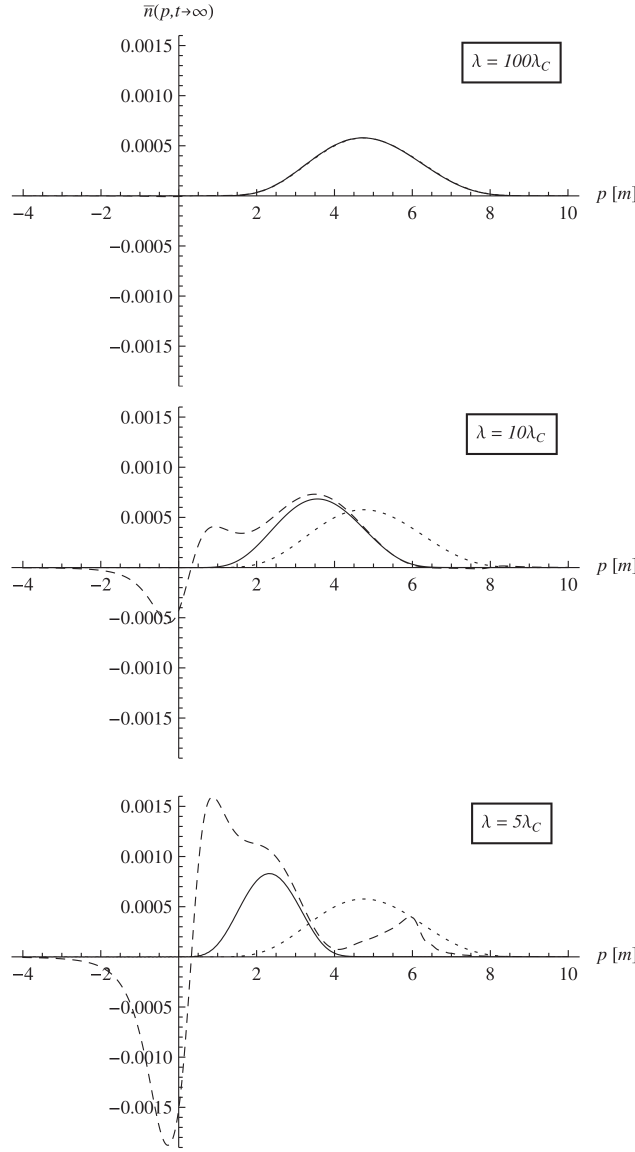

FIG. 1. Comparison of $\bar{n}(p, t \to \infty)$ for the full solution (solid) with the l.o. derivative expansion (dashed) and the local density approximation (dotted) for $\tau = 10/m$, $E_0 = 0.5E_{\mathrm{cr}}$.

For $\lambda \lesssim 4\lambda_C$, however, the height of $\bar{n}(p, t \to \infty)$ decreases again as more and more field excitations gain too little energy in order to finally turn into real particles. For $\lambda = \lambda_C$, the energy content of the electric field is ultimately so small that none of the vacuum fluctuations eventually turns into real particles. This observation is in good agreement with previous studies of space-dependent electric fields $E(x)$ [15–17]: The pair creation process is expected to terminate once the work done by the electric field over its spatial extent is smaller than twice the electron mass. As the pair creation process occurs at time scales of the order of the Compton time $t_C = 1/m$, which

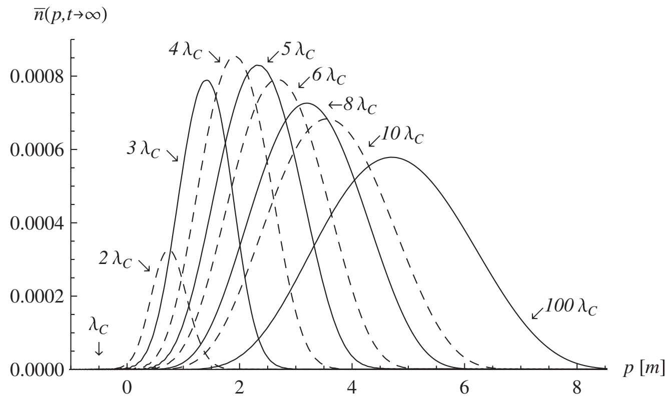

FIG. 2. Comparison of $\bar{n}(p, t \to \infty)$ for $\tau = 10/m$, $E_0 = 0.5E_{cr}$ and different values of λ. Note that $\bar{n}(p, t \to \infty) = 0$ for $\lambda = \lambda_C$.

is smaller than the time scale of the electric pulse $\tau = 10/m$, this estimate should be reasonable in our case as well. The corresponding estimate for the pair creation process to terminate for $E_0 = 0.5E_{cr}$ is in fact in good agreement with our results

$$\lambda < \frac{E_{cr}}{E_0}\sqrt{\frac{2}{\pi}}\lambda_C \simeq 1.6\lambda_C. \qquad (25)$$

Number of created particles.—In Fig. 3 we compare the asymptotic value $\bar{N}(t \to \infty)$ obtained from the full solution with the leading order derivative expansion as well as with the local density approximation for different values of λ. Again, we observe good agreement between the full solution and the various approximations for large λ, however, substantial deviation for small λ. Most notably, only the full solution shows the sharp drop of $\bar{N}(t \to \infty)$ for small λ in accordance with Eq. (25).

Conclusions.—We have presented an *ab initio* real-time calculation of the Schwinger effect in the presence of a simple space- and time-dependent electric field pulse in 1 + 1 dimensions, showing various remarkable features:

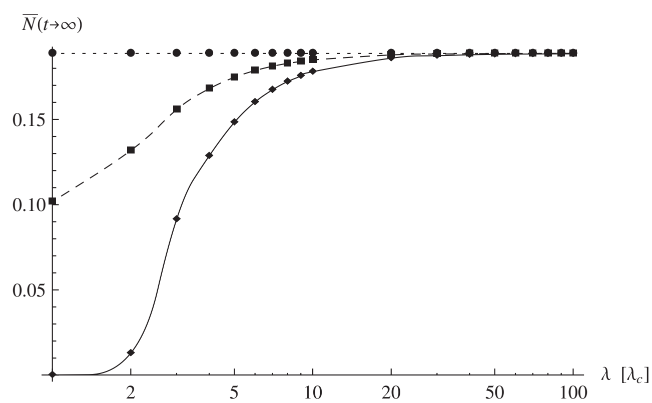

FIG. 3. Comparison of $\bar{N}(t \to \infty)$ for the full solution (solid) with the l.o. derivative expansion (dashed) and the local density approximation (dotted) for $\tau = 10/m$, $E_0 = 0.5E_{cr}$.

Most notably, we observe a new self-bunching effect in phase space which can naturally be interpreted in terms of the space and time evolution of the quantum excitations. The pair creation process eventually terminates for spatially small pulses once the work done by the electric field is too small in order to provide the rest mass of an electron-positron pair. Whereas the derivative expansion is quantitatively able to signal these self-bunching effects, the local density approximation fails to describe these important properties.

These results suggest further studies of the Schwinger effect in realistic space- and time-dependent electric fields in 3 + 1 dimensions. The goal is to consistently describe the Schwinger effect beyond the mean field level by taking into account photon corrections to the background electric field and subsequent collision and radiation processes. In the long run, we expect the self-bunching effect to play an important role in the generation of tailored electron or positron beams.

F. H is supported by the DOC program of the ÖAW and by the FWF doctoral program DK-W1203. H. G acknowledges support by the DFG through grants SFB/TR18, and Gi 328/5-1.

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