Holographic Schwinger Effect

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We study tunneling pair creation of W bosons by an external electric field on the Coulomb branch of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We use AdS/CFT holography to find a generalization of Schwinger's formula for the pair production rate to the strong coupling, planar limit which includes the exchange of virtual massless particles to all orders. We find that the pair creation formula has an upper critical electric field beyond which the process is no longer exponentially suppressed. The value of the critical field is identical to that which occurs in the Born-Infeld action of probe $D3$-branes in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, where AdS_5 and S^5 are 5-dimensional anti–de Sitter space and the 5-sphere, respectively.

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One of the interesting attributes of string theory is the existence of an upper critical electric field [1,2]. Opposite electric charges reside at the end points of open strings. An electric field pulls them in opposite directions. When the field exceeds the string tension, the barrier to stretching the string disappears and strings are unstable. One might wonder, especially in light of AdS/CFT holography, whether this phenomenon can also be visible in a quantum field theory.

In this Letter, we shall address this issue in the quantum field theory for which holography is most firmly established, the superconformal $\mathcal{N} = 4$ Yang-Mills theory. There, we can find an electric field by studying the theory on the Coulomb branch, with gauge group $U(N + 1)$ spontaneously broken by a vacuum expectation value of the scalar fields to $U(1) \times U(N)$. We then consider an electric field of the $U(1)$ gauge theory. It would act on the massive W bosons which have $U(1)$ charges $\pm g_{\mathrm{YM}}$ (the Yang-Mills coupling constant) and transform in the fundamental representation of the residual gauge group $U(N)$. The W bosons are $\frac{1}{2}$-BPS particles and form a short multiplet of the $\mathcal{N} = 4$ supersymmetry algebra which contains scalar, spinor, and vector fields. We will study a dynamical process, the Schwinger effect of pair production for $W^{\pm}$ bosons in a constant electric field.

Before we discuss holography, let us consider a simple field theory argument as to why there could be an upper critical electric field. In order to become real particles, a virtual particle-antiparticle pair that is created by a vacuum fluctuation must gain an energy equal to its combined rest masses $2m$ (we set $c = 1 = \hbar$). This energy could be supplied by an electric field where the pair is pulled in opposite directions. Upon separating by a distance d, the pair gains energy Ed and becomes physical particles when $d \sim 2m/E$. This process is tunneling through a barrier of height $\sim 2m$ and width $\sim 2m/E$. The amplitude should therefore be suppressed by an exponential of the product, $\sim \frac{m^2}{E}$. Exponential suppression with this quantity in the exponent indeed appears in the formula for the tunneling probability which was computed long ago by Schwinger [3] and is quoted in Eq. (6) below. We shall be interested in how Coulomb interactions would modify this effect. With a Coulomb interaction added, the tunneling barrier has profile $V_{\mathrm{eff}}(d) = 2m - Ed - \frac{\alpha}{d}$, where α contains the electric charge.

This barrier is shown in Fig. 1 for different values of the electric field. If the field is sufficiently small, it is positive in a certain range of distances, and the asymptotic region where the particles can be on-shell is separated from the origin, where the pair is created, by a potential barrier. Consequently, at small fields the pair creation is a tunneling

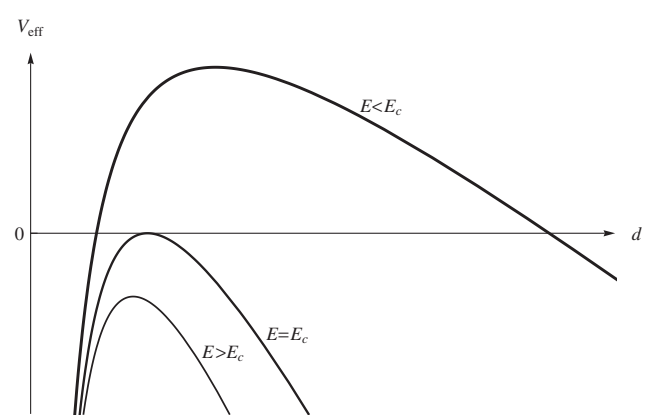

FIG. 1. The energy barrier V_{eff} to pair creation is plotted versus the pair separation d for various values of the electric field.

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process and its amplitude is exponentially suppressed. However, when the field reaches the critical value, the potential is negative everywhere and pair creation does not require tunneling. Its probability is no longer exponentially suppressed. The $\mathcal{N} = 4$ Yang-Mills theory that we are interested in is indeed in a Coulomb phase and the effective coupling of the planar limit of the theory at large 't Hooft coupling, $\lambda = g_{\rm YM}^2 N$, deduced from an AdS/CFT computation of the W boson Wilson loop for parallel lines in Ref. [4] is $\alpha = \frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(\frac{1}{4})}$. An estimate of the critical field is

$$E_c \sim \frac{m^2}{\alpha} = \frac{\Gamma^4(\frac{1}{4})m^2}{4\pi^2\sqrt{\lambda}} \approx 0.70\frac{2\pi m^2}{\sqrt{\lambda}}, \tag{1}$$

which is remarkably similar to one which we shall find in (5) below using holography. This simple argument underestimates it by about 30%.

In the IIB string theory, which is the holographic dual of $\mathcal{N} = 4$ Yang-Mills theory, the breaking of $U(N+1)$ by the Higgs mechanism to $U(1) \times U(N)$ is obtained by separating one $D3$-brane from a parallel stack of N coincident $D3$-branes. W bosons are open strings stretched between the separated $D3$-brane and the stack. In the large N limit, the stack of $D3$-branes is replaced by the background spacetime $\mathrm{AdS}_5 \times \mathrm{S}^5$ which is the product of 5-dimensional anti–de Sitter space AdS_5 and the 5-sphere S^5 and has metric $dS^2 = L^2(r^2 dx_\mu dx^\mu + \frac{dr^2}{r^2} + d\Omega_5^2)$ and N units of the Ramond-Ramond 4-form flux. The radius of curvature L of the background is related to the Yang-Mills theory 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ and the fundamental string scale $\ell_s = \sqrt{2\pi\alpha'}$ by $L^2 = \sqrt{\lambda}\ell_s^2/2\pi$. The probe $D3$-brane fills the four space-time dimensions x^μ and sits at a fixed AdS radius r_0 and at a point on S^5 [given by a unit 6-vector $\hat{n}$ in (10) below]. In the large N planar limit, where N is set to infinity holding λ fixed, and then in the large λ limit, the probe $D3$-brane can be treated as a classical object embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [5]. It has a world-volume metric $ds^2 = L^2 r_0^2 dx_\mu dx^\mu$ and a $U(1)$ world-volume gauge symmetry. If we turn on an electric field E on the world volume of the probe brane, it is described by the Dirac-Born-Infeld and Wess-Zumino actions [8],

$$S = T_3 \int d^4x \left[-\sqrt{-\det(g_{\mu\nu} - \ell_s^2 F_{\mu\nu})} + \omega^{(4)} \right], \tag{2}$$

$$= \frac{L^4 r_0^4}{2\pi \ell_s^4 g_s} \int d^4x \left(-\sqrt{1 - \frac{\ell_s^4}{L^4 r_0^4} E^2} + 1 \right). \tag{3}$$

Here, T_3 is the $D3$-brane tension, $\omega^{(4)}$ is the Ramond-Ramond 4-form, and $g_s = g_{\rm YM}^2/4\pi$ is the closed string coupling constant. For a flat brane on flat space, $g_{\mu\nu}$ in (2) would be replaced by the flat Minkowski space metric diag$(-1, 1, 1, 1)$ and (2) would indeed be singular for an electric field where the argument of the square root vanishes, $E_{\rm crit.}^{\rm flat} = \frac{1}{\ell_s^2} = \frac{1}{2\pi\alpha'}$, equal to the flat space string tension, as expected. In the AdS background, the metric in (2) has a warp factor, the appropriate action is (3), and it also has a critical field, $E_c = L^2 r_0^2/\ell_s^2$. It is interesting to write this critical field using Yang-Mills theory parameters. The parameter r_0 is related to the mass of the W boson, which is given by the energy of a fundamental string that is suspended between the probe brane at $r = r_0$ and the Poincarè horizon at $r = 0$. The world sheet metric of such a string would be $d\sigma^2 = L^2(-r^2 dt^2 + \frac{dr^2}{r^2})$ and the mass is

$$m = \frac{1}{\ell_s^2} \int_0^{r_0} dr \sqrt{-|g(t, r)|} = \frac{L^2}{\ell_s^2} \int_0^{r_0} dr = \frac{\sqrt{\lambda} r_0}{2\pi}. \tag{4}$$

Using (4) to eliminate r_0 from our expression for the critical electric field, we conclude that

$$E_c = \frac{2\pi m^2}{\sqrt{\lambda}}. \tag{5}$$

Duality then indicates that the large N and large λ limit of $\mathcal{N} = 4$ Yang-Mills theory on its Coulomb branch also has this critical field, which should be compared with (1). In the following, we will examine the field theory manifestation of this critical field in more detail.

Schwinger's formula [3] for the probability of the production of charged particle-antiparticle pairs, $P = 1 - e^{-\gamma v}$, where the contribution to γ from a particle with spin j and mass m in a constant electric field E over space-time volume v is

$$\gamma_j = \frac{(2j+1)E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)(2j+1)}}{n^2} e^{-[(\pi m^2 n)/|E|]}. \tag{6}$$

This formula applies to a weak coupling limit where radiative corrections from the emission and reabsorption of virtual particles are neglected. It can be computed, as Schwinger originally did, from a proper time representation of the appropriate Feynman diagrams. Alternatively, it can be found (for a spin 0 particle) from the imaginary part of the Euclidean worldline path integral

$$\gamma_0 = -\frac{2}{v} \Im \int_0^\infty \frac{dT}{T} \int \mathcal{D}x_\mu e^{-\int_0^1 d\tau[(\dot{x}^2/4T) + m^2 T - iA_\mu \dot{x}^\mu]}, \tag{7}$$

with periodic boundary conditions $x_\mu(\tau + 1) = x_\mu(\tau)$. For a constant field, $A_\mu = -\frac{1}{2}F_{\mu\nu}x_\nu$. The nonzero components of $F_{\mu\nu}$ are $F_{12} = -F_{21} = -iE$, the "i" is a result of analytic continuation to Euclidean space, and we shall assume $E > 0$. As the exponentials in the summand in (6) suggest, (7) can be computed as a sum over instanton amplitudes for tunneling through the potential barrier of pair creation [9]. In Euclidean space, the electric field acts as a magnetic field and the instanton of the Euclidean worldline path integral is a cyclotron orbit of the charged particle, that is, a circle trajectory. The integer n, which is summed in (6), is the number of instantons, which is the number of times the particle traverses the circle. With the

solution of the equation of motion for T, $T = \frac{1}{2m}\sqrt{\int \dot{x}^2}$, and the circle ansatz, $x_{0\mu} = R\hat{r}$ where $\hat{r} = (\cos(2\pi n\tau), \times \sin(2\pi n\tau), 0, 0)$, $n = 1, 2, \ldots$, the classical action is

$$S_{\mathrm{cl}(0)} = 2\pi nRm - \frac{2\pi nER^2}{2}. \tag{8}$$

The classical equation of motion for x_μ is solved when R is adjusted to an extremum of (8), $R = m/E$, which is a maximum. Fluctuations of R are tachyonic and integrating them would produce the factor of "i" which gives the Euclidean path integral an imaginary part. Substituting $R = m/E$ into (8) yields $S_{\mathrm{cl}(0)} = \frac{\pi m^2}{E} n$ which is identical to the exponent in Schwinger's formula, Eq. (6). To determine the prefactor of the exponential in (6) it is necessary to analyze fluctuations about the classical solution. Reference [9] showed how to get the prefactor of the $n = 1$ term by doing the quadratic integral over fluctuations. Though it would be desirable to do so here, for example, to understand the nature of the amplitude when exponential suppression is absent, we will not address this interesting problem in the present paper, but will reserve it for a more detailed exposition elsewhere.

In the planar limit of $\mathcal{N} = 4$ Yang-Mills theory, Schwinger's formula (6) applied to W bosons would be modified in a number of ways. It will have an overall factor of N to reflect the number of W bosons and it comes from the vacuum energy with one W-boson loop. Contributions with additional W-boson loops are suppressed by factors of $1/N$ and can be ignored in the large N limit. As well, contributions from virtual $U(1)$ photons are proportional to $g_{\mathrm{YM}}^2 = \lambda/N$ and are suppressed at large N. Interactions with the massless particles of the unbroken $U(N)$ gauge theory are ignored in the weak coupling limit which produces (6), but must be included as, in the large N limit, planar Feynman diagrams will contribute at all orders in λ. For a scalar field in the W-boson supermultiplet, these contributions (as well as the overall factor of N) can be taken into account by adding a Wilson loop amplitude to the path integral (7). The action in (7) becomes

$$S = \int_0^1 d\tau\left(\frac{\dot{x}^2}{4T} + m^2T + \frac{iF_{\mu\nu}}{2}x_\mu\dot{x}_\nu\right) - \ln W[x_\mu], \tag{9}$$

where $W[x_\mu]$ is the Wilson loop. The appropriate quantity (in the large W-mass limit) is [4,10,11]

$$W[x] = \langle \mathrm{Tr}\mathcal{P}e^{\int_0^T d\tau(i\dot{x}_\mu A_\mu + |\dot{x}|\hat{n}_I\Phi_I)}\rangle. \tag{10}$$

Here, $\hat{n}_I$ is a unit vector in the direction of the scalar field condensate $\langle\Phi_I\rangle$. The gauge field A and scalar Φ_I transform in the adjoint representation of $SU(N)$ and the trace over $SU(N)$ indices is of order N. The path integral with action (9) is semiclassical when the mass of the W boson is large, $m^2 \gg E$. We shall also consider strong coupling, $\lambda \gg 1$. These limits are compatible with electric fields in the range $E \sim \frac{m^2}{\sqrt{\lambda}}$ where we expect to find a critical field. The conformal symmetry of $\mathcal{N} = 4$ Yang-Mills theory implies that, when evaluated on a circle, $x = R\hat{r}$, W is a function of mR and rotation symmetry implies $\frac{\delta}{\delta x^\mu}W[x]|_{x=R\hat{r}} = \hat{r}_\mu \frac{d}{dR}W$. Consequently, once the radius is adjusted to an extremum of the action, now including the Wilson loop, the circle is still a solution of the classical equation of motion derived from (9). Moreover, for the infinite mR limit, exact results for $W[\mathrm{circle}]$ [12–14] and an expression for quadratic fluctuations about a straight line [15,16] which can easily be adapted to a circle are available. Indeed, the known strong coupling behavior for a circle (wrapped n times), $\ln W \sim n\sqrt{\lambda}$, combined with (8) would lead to a corrected classical action $S_{\mathrm{cl}(1)} = (\frac{\pi m^2}{E} - \sqrt{\lambda})n$ which suggests a critical behavior at large λ where $S_{\mathrm{cl}(1)}$ goes to zero and the sum over n in the Schwinger amplitude would no longer be exponentially suppressed. However, the computation of the Wilson loop we are using is already specialized to infinite $mR \sim \frac{m^2}{E}$. [A symptom of the problem is the fact that the critical E that we would estimate from the discussion above differs from (5) by a factor of 2.]

To correctly estimate $\ln W$, we shall need the expectation value of the appropriate loop with a large but finite W mass. For this we return to the probe $D3$-brane placed at radius r_0 in AdS_5 and replace the action in (8) by the disc amplitude for a string which intersects a probe $D3$-brane on the circle, $x(\tau) = R\hat{r}$ and couples to the electric field at the boundary of its world sheet. In the large λ limit, the string sigma model is semiclassical and the problem reduces to finding a disc of extremal area. The sigma model action in the conformal gauge is

$$S_{\mathrm{st}} = \frac{L^2}{2\ell_s^2}\int_0^1 d\tau\int_{\sigma_0}^\infty d\sigma\left(r^2\partial X_\mu\bar{\partial}X_\mu + \frac{\partial r\bar{\partial}r}{r^2}\right) + i\oint A, \tag{11}$$

where $(\partial, \bar{\partial}) = (\partial_\sigma \pm i\partial_\tau)$. The last term is the coupling of the boundary of the string world sheet to the gauge field. The equations of motion, Virasoro constraints, and boundary conditions are

$$\partial\bar{\partial}r = r^3\partial X\bar{\partial}X + \frac{1}{r}\partial r\bar{\partial}r, \qquad \partial_a(r^2\partial^a X_\mu) = 0, \tag{12}$$

$$r^2(\partial X)^2 + \frac{(\partial r)^2}{r^2} = 0, \qquad r^2(\bar{\partial}X)^2 + \frac{(\bar{\partial}r)^2}{r^2} = 0, \tag{13}$$

$$X(\tau, \sigma_0) = R\hat{r}(\tau), \qquad r(\tau, \sigma_0) = r_0. \tag{14}$$

They have the solution

$$X = \frac{\cosh(2\pi n\sigma_0)}{\cosh(2\pi n\sigma)}R\hat{r}, \qquad r = r_0\frac{\tan(2\pi n\sigma_0)}{\tanh(2\pi n\sigma)}, \tag{15}$$

when $\sinh(2\pi n\sigma_0) = 1/Rr_0$. We then replace (8) with the on-shell string action, found by substituting (15) into (11) and using (4) to get [17]

$$S_{\mathrm{cl}(2)} = n\left(\sqrt{(2\pi mR)^2 + \lambda} - \sqrt{\lambda}\right) - \frac{1}{2}(2\pi n)ER^2. \quad (16)$$

This expression should be accurate when mR is large and when λ is large. It reproduces (8), corrected by the Wilson loop term $-n\sqrt{\lambda}$ in the limit where $mR \gg \sqrt{\lambda}$. The radius should now be fixed to an extremum of (16),

$$R = \frac{1}{2\pi m}\sqrt{\left(\frac{2\pi m^2}{E}\right)^2 - \lambda}. \quad (17)$$

There is a critical value of the electric field where this radius shrinks to zero, given by the value of the critical field E_c in (5). The classical action

$$S_{\mathrm{cl}(2)} = \frac{n\sqrt{\lambda}}{2}\left(\sqrt{\frac{E_c}{E}} - \sqrt{\frac{E}{E_c}}\right)^2 \quad (18)$$

also vanishes when E approaches E_c and the summation over n is unsuppressed. Moreover, it agrees with the Schwinger result in the weak field $E \ll E_c$ limit.

The world sheet in (15) can be continued to Lorentzian signature where it is the locus of $-t^2 + x^2 + 1/r^2 = R^2 + 1/r_0^2$ with $r \leq r_0$, that is part of AdS_2. At any fixed time, t, the profile of the string is

$$x(t, r) = \pm\sqrt{t^2 + R^2 + \frac{1}{r_0^2} - \frac{1}{r^2}} \quad (19)$$

and is depicted in Fig. 2. The end points on the probe brane at $r = r_0$ are the position of the particle and the antiparticle $\pm\sqrt{t^2 + R^2}$. At $t = 0$, they are separated by a distance $2R$. After the initial time, the particle and antiparticle follow trajectories with a constant proper acceleration of magnitude $a = \frac{1}{R}$ and in opposite directions. If they were simple charged particles with mass m, the electric field would give them proper acceleration $a = \frac{E}{m}$, agreeing with the result $R = \frac{m}{E}$ for the radius that was found by extremizing the action (8). When we extremize the stringy action (16) to get (17), the radius has decreased, so the proper acceleration is greater, $a = \frac{E/m}{\sqrt{1-(E/E_c)^2}}$. The end point of the string in a strong electric field seems to have less inertia than a particle would have. Remember that a particle is a string which hangs from the probe brane to the Poincaré horizon, whereas the appropriate string for our problem here, shown in Fig. 2, never reaches the Poincarè horizon; rather, it lags behind the end point and it joins with the string of the antiparticle. This join persists regardless of the separation. (There is no Gross-Ooguri phase transition [19] in this case.) This joining of the string is responsible for the classical action in (16) which is slightly smaller an the analogous one for a relativistic particle given in (8). In the field theory language, the effect can be seen as coming from the λ dependence of the action and it reflects the negative Coulomb interaction energy of the particle and antiparticle, which persists at strong coupling, and was the basis for our argument leading to Eq. (1).

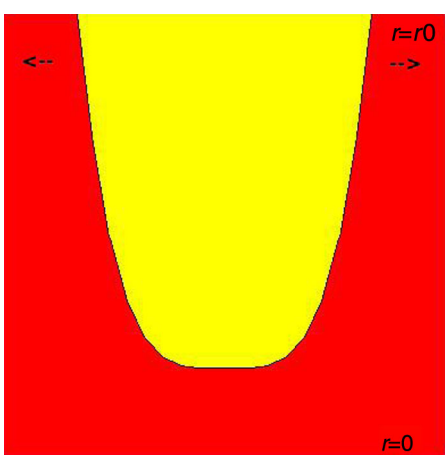

FIG. 2 (color online). String profile. The end points of the string at $r = r_0$ have constant proper acceleration in opposite directions.

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