Measurement of the Noise Spectrum Using a Multiple-Pulse Sequence

Tatsuro Yuge,[1,*] Susumu Sasaki,[2] and Yoshiro Hirayama[3,4]

[1]*IIAIR, Tohoku University, Sendai 980-8578, Japan*
[2]*Department of Materials Science and Technology, Niigata University, Niigata 950-2181, Japan*
[3]*Department of Physics, Tohoku University, Sendai 980-8578, Japan*
[4]*ERATO Nuclear Spin Electronics Project, Sendai 980-8578, Japan*

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A method is proposed for obtaining the spectrum for noise that causes the phase decoherence of a qubit directly from experimentally available data. The method is based on a simple relationship between the spectrum and the coherence time of the qubit in the presence of a π pulse sequence. The relationship is found to hold for every system of a qubit interacting with the classical-noise, bosonic, and spin baths.

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Introduction.—Implementation of quantum information processing (QIP) requires maintaining the quantum coherence of a system during its operation. In reality, however, systems are not completely isolated from the environment. Interactions with the environment are system noise and cause decoherence. One of the challenges in QIP implementation is to control the system appropriately in a noisy environment and/or to reduce the noise level by weakening the system-environment coupling. Strategies for these approaches have been developed, including quantum error correction [1–3], quantum estimation [4,5], and dynamical decoupling (DD) [6–17].

In DD, the application of multiple π pulses cancels out the noise and effectively suppresses the system-environment coupling [6]. This basic idea of DD comes from concepts in pulsed nuclear magnetic resonance (NMR) [18]. Experiments have demonstrated that decoherence can be suppressed by using methods common to pulsed NMR, such as alternating-phase-Carr-Purcell (APCP) and Carr-Purcell-Meiboom-Gill (CPMG) methods [19,20]. Subsequent theoretical [7–14] and experimental [15–17] studies on DD in various kinds of spin- and charge-related qubit systems other than NMR compared several DD methods and suggested that optimization of DD requires knowledge of the noise properties.

For any strategy (not only DD), knowledge of the noise properties is helpful because it can be used to improve the strategy. It is therefore important to identify the noise properties of the environment. For longitudinal decoherence (energy relaxation) noise, the noise spectrum can be obtained from the longitudinal relaxation time (called T_1 in NMR) [18,21,22], and a qubit (two-level system) has been proposed as a noise spectrum analyzer [23]. In contrast, a method for measuring transverse decoherence (pure dephasing) noise has not been established although several methods (also using a qubit) have recently been proposed [10,24,25]. In Ref. [24], it was pointed out that the relationship between noise spectrum and coherence can be used for estimating the spectrum. In Ref. [10], a method was described for obtaining the moments of the spectrum by using the Uhrig pulse sequence [8]. In Ref. [25], a method was described for obtaining the spectrum at the Rabi frequency by using a nearly continuous and on-resonant control field.

In this Letter we propose a method for measuring the dephasing noise spectrum in which a simple sequence of equidistant π pulses (such as an APCP or CPMG sequence) is used. The spectrum at frequency $\pi/2\tau$, where 2τ is the interval between pulses, is evaluated directly from experimentally obtained values of the coherence times by using a relationship between the noise spectrum and the coherence time for a sequence of a sufficiently large number of pulses. We show that this relationship holds for the classical-noise, spin-boson, and spin-spin bath models.

Model and noise spectrum.—We use a model of a single qubit (spin $S = 1/2$) interacting with the environmental degrees of freedom (bath). The Hamiltonian of the total system is given by

$$\hat{H} = \frac{\hbar}{2}(\Omega + \hat{\xi})\hat{\sigma}_z + \hat{H}_{\rm bath}, \tag{1}$$

where $\hat{\xi}$ and $\hat{H}_{\rm bath}$ are the bath operators. The Pauli matrices of the qubit are denoted by $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$. This is a pure dephasing model because the interaction Hamiltonian $\hat{H}_{\rm int} = (\hbar/2)\hat{\xi}\hat{\sigma}_z$ between the qubit and bath includes only $\hat{\sigma}_z$. In other words, there is no energy relaxation in this model (T_1 is infinite in NMR terminology). The spin (qubit) is subject to a static magnetic field Ω and a random magnetic field (noise) $\hat{\xi}$ in the z direction; the random field $\hat{\xi}$ is generated by the bath.

There are several variations of the model depending on the nature of the bath. Here we consider three of them. The first is a classical-noise model [10], where ξ is a stationary Gaussian stochastic process (classical random variable) with zero mean. (In this case, $\hat{H}_{\rm bath}$ does not appear.) The second one is a spin-boson model [6,8,9] in which

$\hat{\xi} = \sum_j \lambda_j(\hat{b}_j^\dagger + \hat{b}_j)$ and $\hat{H}_{\text{bath}} = \sum_j \hbar\omega_j \hat{b}_j^\dagger \hat{b}_j$, where $\hat{b}_j$ ($\hat{b}_j^\dagger$) is the jth mode annihilation (creation) operator of the bosonic bath, and λ_j is the coupling strength between the qubit and the jth mode boson. The third one is a spin-spin bath model [26] in which $\hat{\xi} = \sum_j \mu_j(\hat{s}_+^j + \hat{s}_-^j)$ and $\hat{H}_{\text{bath}} = \sum_j (\hbar/2)\omega_j \hat{s}_z^j$, where $\hat{s}_+^j = \hat{s}_x^j + i\hat{s}_y^j$ and $\hat{s}_-^j = \hat{s}_x^j - i\hat{s}_y^j$. Here, $\hat{s}_x^j$, $\hat{s}_y^j$, and $\hat{s}_z^j$ are, respectively, the x, y, and z components of the Pauli matrices of the jth spin in the bath, and μ_j is the coupling strength between the qubit (spin of interest) and the jth spin (in the bath).

The noise spectrum is defined as the symmetrized power spectral density function of the random field:

$$S(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{1}{2} \langle \check{\xi}(t)\check{\xi}(0) + \check{\xi}(0)\check{\xi}(t)\rangle_{\text{bath}}. \quad (2)$$

In the classical-noise model, $\langle\cdots\rangle_{\text{bath}}$ is the average over the stochastic variable [$\check{\xi}(t) = \xi(t)$]. In the spin-boson and spin-spin bath models, $\check{\xi}(t) = e^{i\hat{H}_{\text{bath}}t/\hbar}\hat{\xi}e^{-i\hat{H}_{\text{bath}}t/\hbar}$, and $\langle\cdots\rangle_{\text{bath}} = \text{Tr}_{\text{bath}}(\hat{\rho}_{\text{bath}}^{\text{eq}}\cdots)$ is the average in an equilibrium state of the bath ($\hat{\rho}_{\text{bath}}^{\text{eq}} = e^{-\beta\hat{H}_{\text{bath}}}/Z_{\text{bath}}$).

Pulse sequence and generalized coherence time.—The density matrix (which describes the state) of the qubit is denoted by $\hat{\rho}_s$. In the classical-noise model, $\hat{\rho}_s$ is given by the average of $\hat{\hat{\rho}}_s$ over the stochastic variable: $\hat{\rho}_s = \langle\hat{\hat{\rho}}_s\rangle_{\text{bath}}$, where $\hat{\hat{\rho}}_s$ is the qubit state with one realization of the stochastic variable. In the spin-boson and spin-spin bath models, $\hat{\rho}_s$ is given by the partial trace of the density matrix $\hat{\rho}$ of the total system: $\hat{\rho}_s = \text{Tr}_{\text{bath}}\hat{\rho}$.

The coherence is quantified using a nondiagonal component of the density matrix of the qubit: $\rho_{s,+-} = \langle +|\hat{\rho}_s|-\rangle$. Here, $|\pm\rangle$ is the eigenstate of $\hat{\sigma}_z$ corresponding to the eigenvalue ± 1. Experimentally, the coherence is measured using the transverse (x and y) components of the qubit (spin) and the relations $\langle\hat{\sigma}_x\rangle = 2\text{Re}\rho_{s,+-}$ and $\langle\hat{\sigma}_y\rangle = -2\text{Im}\rho_{s,+-}$.

Now we consider a situation in which ideal (instantaneous) π pulses (about the x or y axis) are repeatedly applied to the qubit. As shown in Fig. 1(a), the pulses are applied at times $t_1, t_2, \ldots, t_n$, where n is the total number of pulses. After the application of the sequential pulses, we measure the normalized coherence at time t ($> t_n$): $W(t) = |\rho_{s,+-}(t)|/|\rho_{s,+-}(0)|$.

When we apply a sufficiently large number of pulses (keeping the minimal interpulse time fixed, so that t is also sufficiently large), the coherence exhibits an exponential decay (as shown in the models later):

$$W(t) \sim \exp(-t/T_2^L). \quad (3)$$

This time dependence enables us to define uniquely the coherence time T_2^L for a multiple-pulse sequence, which we call "generalized" coherence time (for a reason mentioned later). T_2^L differs from conventional coherence time T_2^{SE}, which is defined using spin echo (SE) experiments ($T_2^L > T_2^{\text{SE}}$ in most cases). Generally, it can be shown that T_2^L depends on the pulse sequence.

In the three models, the time evolution of the normalized coherence in the presence of a pulse sequence can be expressed as (weak coupling condition $\mu_j \ll \omega_j$ is necessary for the spin-spin bath model) [8–10,26,27]

$$W(t) = \exp\left(-\int_0^\infty \frac{d\omega}{2\pi} S(\omega)|\tilde{f}_t(\omega)|^2\right), \quad (4)$$

$$\tilde{f}_t(\omega) = \int_{-\infty}^{\infty} dt' e^{i\omega t'} f_t(t'), \quad (5)$$

$$f_t(t') = \sum_{k=0}^{n} (-1)^k \Theta(t_{k+1} - t')\Theta(t' - t_k), \quad (6)$$

where $t_0 = 0$, $t_{n+1} = t$, and $\{t_k\}_{k=1}^n$ is the set of times when the pulses are applied, and $\Theta(T)$ is the Heaviside step function. As shown in Fig. 1(b), $f_t(t')$ depends on the pulse sequence and takes a nonzero value ($+1$ or -1) only for $0 < t' < t$.

Relationship between T_2^{SE} and S.—Before presenting the main results, we show the relationship between T_2^{SE} and S. For the SE pulse sequence ($\pi/2(x) - \tau - \pi(x) - \tau$–signal), we can easily find that $|\tilde{f}_t(\omega)|^2 = (16/\omega^2)\sin^4(\omega\tau/2)$. Substituting this into Eq. (4) and using the asymptotic behavior $(1/\omega^2)\sin^4(\omega\tau/2) \sim (\pi/4)\delta(\omega)\tau$ as $\tau \to \infty$, we get the asymptotic τ dependence of the SE coherence $W^{\text{SE}}(2\tau)$,

$$W^{\text{SE}}(2\tau) \sim \exp\left(-\frac{1}{2}S(0)2\tau\right) \quad \text{as } \tau \to \infty, \quad (7)$$

which yields the relationship between T_2^{SE} and S:

$$\frac{1}{T_2^{\text{SE}}} = \frac{1}{2}S(0). \quad (8)$$

Although this formula is common in the field of NMR [18,28], it has a significant implication; we should use the asymptotic exponential decay of the SE coherence to define T_2^{SE}, as seen in Eq. (7) [29,30]. This is contrastive to the usual evaluation of coherence time in SE experiments, in which the behaviors of the SE coherence at smaller τ are generally used. Their use would not provide a unique definition of coherence time because the functional form of the SE coherence at smaller τ depends on the systems. [In some systems exponential decays are observed with

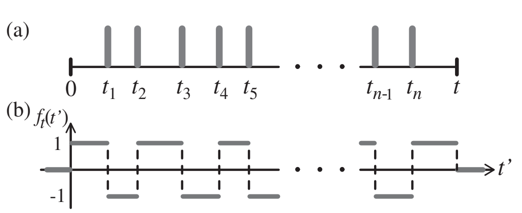

FIG. 1. (a) Illustration of a π pulse sequence. (b) Function $f_t(t')$ corresponding to illustration (n: even).

certain time constants (different from $T_2^{\rm SE}$ in some cases), while in other systems Gaussian decays are observed with certain time constants (different from $T_2^{\rm SE}$).]

Relationship between T_2^L and S.—In the long time limit (large n limit) keeping the minimal interpulse time fixed, we get the power spectral density function of $f_t(t')$: $I(\omega) \equiv \lim_{t\to\infty}(1/t)|\tilde{f}_t(\omega)|^2$. Hence $|\tilde{f}_t(\omega)|^2 \sim I(\omega)t$ as $t \to \infty$. From this and Eq. (4), we find that the asymptotic behavior of the coherence is an exponential decay:

$$W(t) \sim \exp\left(-\int_0^\infty \frac{d\omega}{2\pi} S(\omega)I(\omega)t\right) \quad \text{as } t \to \infty. \tag{9}$$

The necessary number of pulses to extract this asymptotic exponential decay is independent of the noise spectrum because $f_t(t')$ is determined only by the pulse sequence. Comparing this equation with the definition of T_2^L [Eq. (3)], we obtain

$$\frac{1}{T_2^L} = \int_0^\infty \frac{d\omega}{2\pi} S(\omega)I(\omega). \tag{10}$$

If the sequence consists of equidistant pulses with $t_{k+1} - t_k = 2\tau$ ($k = 1, 2, \ldots, n-1$), $f_t(t')$ is a periodic function of t' with a period of 4τ (for $0 < t' < t$). In this case, the power spectral density function $I(\omega)$ (with fixed τ) becomes

$$I(\omega) = 2\pi \sum_{m=-\infty}^{\infty} |C_m|^2 \delta(\omega - \omega_m), \tag{11}$$

where $\omega_m = m\pi/2\tau$, and C_m is the Fourier coefficient of $f_t(t')$: $C_m = (1/4\tau)\int_0^{4\tau} dt' e^{i\omega_m t'} f_t(t')$. Substituting Eq. (11) into Eq. (10), we obtain

$$\frac{1}{T_2^L} = \sum_{m=0}^{\infty} |C_m|^2 S(\omega_m). \tag{12}$$

As an example, we investigate the use of the APCP pulse sequence: $\pi/2(x) - \{\tau - \pi(x) - 2\tau - \pi(\bar{x}) - \tau\}^{n/2}$–signal ($n$: even). For this pulse sequence,

$$f_t(t') = \begin{cases} 1 & \text{for } 4k\tau < t' < (4k+1)\tau \\ & \text{and } (4k+3)\tau < t' < 4(k+1)\tau, \\ -1 & \text{for } (4k+1)\tau < t' < (4k+3)\tau, \\ 0 & \text{otherwise,} \end{cases}$$

where $k = 0, 1, 2, \ldots, n-1$. This yields $|C_m|^2 = (4/\pi^2 m^2)\delta_{m,2l+1}$ ($l = 0, 1, 2, \ldots$). Hence we finally obtain

$$\frac{1}{T_2^L} = \frac{4}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} S(\omega_{2l+1}). \tag{13}$$

Note that we obtain the same result for the Carr-Purcell and CPMG sequences because we assume ideal (instantaneous) pulses. [To avoid pulse-error accumulation in actual experiments (with nonideal pulses), one should use CPMG or APCP.] Since the factor $1/(2l+1)^2$ is smaller for larger l, we can approximate the above equation into

$$\frac{1}{T_2^L} \simeq \frac{4}{\pi^2} S(\pi/2\tau), \tag{14}$$

if $S(\omega)$ rapidly decreases as ω increases. Equations (13) and (14) are the main results of this Letter.

These relationships are qualitatively explained as follows. In many systems, coherence time is dominated by the lowest-frequency component of the noise spectrum (fluctuation-dissipation relation). The pulse sequence with time interval $\sim\tau$ cancels out the noise at frequencies lower than $1/\tau$ (dynamical decoupling). Therefore, the noise spectrum around the frequency $1/\tau$ dominantly contributes to the coherence time in the presence of the pulse sequence.

Note that the infinite τ limit of Eq. (13) is consistent with Eq. (8):

$$\lim_{\tau\to\infty} \frac{1}{T_2^L} = \frac{4}{\pi^2} S(0) \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{1}{2} S(0) = \frac{1}{T_2^{\rm SE}}, \tag{15}$$

where we use $\lim_{\tau\to\infty} \omega_{2l+1} = 0, \ \forall\ l$ in the first equality, and $\sum_{l=0}^{\infty} 1/(2l+1)^2 = \pi^2/8$ in the second equality. This equation enables us to interpret T_2^L as a generalization of $T_2^{\rm SE}$ into nonzero frequencies, which is the reason we call T_2^L "generalized coherence time."

Similarly, the infinitesimal τ limit of Eq. (13) is given by

$$\lim_{\tau\to+0} \frac{1}{T_2^L} = \frac{1}{2} \lim_{\omega\to\infty} S(\omega). \tag{16}$$

Measurement of noise spectrum.—Using the approximate relation (14) we evaluate the noise spectrum as follows. For a fixed value of 2τ (interpulse time), we evaluate T_2^L by applying a sufficiently large number of pulses and measuring the asymptotic behavior of the coherence. We repeat this procedure for other fixed values of 2τ. Then, by plotting $1/T_2^L$ against $\pi/2\tau$, we obtain S as a function of ω.

Better evaluation of the spectrum is possible by using the precise relation (13). For example, a functional form $S^{\rm fit}(\omega)$ of the noise spectrum (which includes some fitting parameters) based on the above approximate evaluation is phenomenologically introduced. Then we create a function $F(\pi/2\tau)$ similar to that on the right-hand side of Eq. (13) by summing the phenomenological function $S^{\rm fit}$ up to an appropriate cutoff L ($L \gtrsim 10$ would be sufficient):

$$F(\pi/2\tau) = \frac{4}{\pi^2} \sum_{l=0}^{L} \frac{1}{(2l+1)^2} S^{\rm fit}((2l+1)\pi/2\tau). \tag{17}$$

We finally fit $F(\pi/2\tau)$ to the experimental results ($1/T_2^L$ vs $\pi/2\tau$) to obtain the values of the parameters for $S^{\rm fit}$.

Finally, we estimate the frequency range for this method of noise measurement. To measure experimentally the asymptotic behavior of the coherence, we should apply a large number of pulses before the amplitude becomes too small to detect. Hence, the interpulse time 2τ must be smaller than the $1/e$ decay time T_2 of the coherence (in the presence of the SE pulse sequence). This implies that the lower bound on the frequency $\pi/2\tau$ should be π/T_2. The upper bound is determined by the shortest interpulse time that is experimentally available. The value of this time can be of the same order as that of the pulse duration time τ_p. Hence, the upper bound on the frequency $\pi/2\tau$ should be π/τ_p.

Concluding remarks.—In summary, we have described a method for obtaining the dephasing noise spectrum. The method is simple in the sense that we have only to apply sequences of equidistant π pulses to the qubit (spin). The generalized coherence time, evaluated from the asymptotic exponential decay of the coherence in the presence of a sufficiently large number of pulses, is used for evaluating the spectrum. This method is applicable to a system interacting with several independent noise sources. In this case, the total spectrum is the sum of the individual noise spectra.

This method is expected to be valid in a wide range of systems because we have derived it for three models. The single-qubit noise spectrum plays an important role even in multiqubit systems because it significantly contributes to the dynamics of the systems (this is clearly seen when analyzing with the quantum master equation). Extension of the present study remains a theoretical task. We should analyze a model in which the system-environment coupling is given in a general form and/or in which there are both energy and phase relaxations. The projection operator formalism [31] would be helpful in analyzing a general model.

This method should provide new insights into NMR experiments in condensed matter physics. So far the longitudinal relaxation time T_1 has been successfully used for investigating properties of electrons and nuclear spins in condensed matter. The method presented here should enable the use of the generalized coherence time T_2^L for investigating (other) properties of them. This is because the noise spectrum (evaluated from T_2^L) includes information on the environment, the physical origin of which is electrons, other nuclear spins, and so on. In order to make this application useful, it is important to capture the characteristic structure of the spectrum by analyzing microscopic models that include interactions with nuclear spins (e.g., Fermi contact hyperfine and dipolar couplings). An experimental demonstration of the present method in NMR will be reported elsewhere [32].

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*yuge@m.tohoku.ac.jp

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