

How to Decompose Arbitrary Continuous-Variable Quantum Operations

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We present a general, systematic, and efficient method for decomposing any given exponential operator of bosonic mode operators, describing an arbitrary multimode Hamiltonian evolution, into a set of universal unitary gates. Although our approach is mainly oriented towards continuous-variable quantum computation, it may be used more generally whenever quantum states are to be transformed deterministically, e.g., in quantum control, discrete-variable quantum computation, or Hamiltonian simulation. We illustrate our scheme by presenting decompositions for various nonlinear Hamiltonians including quartic Kerr interactions. Finally, we conclude with two potential experiments utilizing offline-prepared optical cubic states and homodyne detections, in which quantum information is processed optically or in an atomic memory using quadratic light-atom interactions.

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Introduction.—Since the proposal of quantum computation as a generalization of computer science, an important theoretical challenge has been how to decompose an arbitrary gate into a universal set. The corresponding theory of discrete-variable decompositions is very extensive and mostly employs matrix representations of logic gates utilizing matrix decomposition techniques [1]. In contrast to discrete-variable theory, there is no established method to decompose an arbitrary operator in the continuous-variable (CV) regime except the proof-of-principle results on universal gate sets in Refs. [2,3]. In particular, Ref. [2] makes use of an exponential operator approximation and proves that by employing certain elementary gate sets (discussed below) one can derive any operator up to a certain error. However, none of these works intend to present a constructive and efficient decomposition recipe.

The problem of decomposition is intrinsically related to the concept of universality. Universality means having a set of operators that allows you to simulate any operator on a certain Hilbert space through concatenations of the elements of the universal set. For our purpose, achieving universality is then equivalent to decomposing, at least approximately, an arbitrary unitary exponential operator to a set of elementary unitary exponential operators:

$$e^{itH(a,a^\dagger)} = \{e^{it_1H_1(a,a^\dagger)}, e^{it_2H_2(a,a^\dagger)}, \dots, e^{it_NH_N(a,a^\dagger)}\}.$$

Here, a and a^\dagger are annihilation and creation operators, respectively, and $\{H_n\}$ are fixed Hermitian functions of mode operators. The coefficients t_1, t_2, \dots are interaction times of the Hamiltonians and are functions of t . Thus, different concatenations of elements of this set for varying interaction times should enable one to simulate an arbitrary operator. We assume that we have access to arbitrary interaction times for the initial set.

In our setting, there are two important criteria for CV gate decompositions: how systematic and how efficient the

decompositions are. Here we shall derive methods according to these criteria and present a systematic and efficient framework for decomposing any given unitary operator that acts on bosonic modes into a universal set of elementary CV gates. Our general method consists of first expressing operators in terms of linear combinations of commutation operators and then realizing each commutation operator and their combinations through approximations. We will discuss the efficiency of the decompositions and present guidelines to obtain an arbitrary order of error. For this purpose, we employ a powerful technique for obtaining efficient approximations (see supplemental material for details [4]). Throughout, we use the convention $\hbar = 1/2$, i.e., the fundamental commutation relation is $[X, P] = i/2$ with $X \equiv (a^\dagger + a)/2$ and $P \equiv i(a^\dagger - a)/2$.

General Gaussian decompositions.—For Gaussian operators, i.e., second-order operators, exact and finite decompositions to elementary sets are known (here, order is defined as the polynomial order of the mode operators in the Hamiltonian of a given operator). For example, the Bloch-Messiah decomposition allows you to decompose any second-order operator, i.e., any unitary Gaussian operation, to passive linear multimode optics, single-mode squeezing, and displacement operations [5].

In Ref. [3] the following set is presented as a single-mode Gaussian universal set: $\{e^{i(\pi/2)(X^2+P^2)}, e^{it_1X}, e^{it_2X^2}\}$, and in Ref. [6], similar to the Bloch-Messiah decomposition, a recipe is given to decompose any single-mode transformation of second order to this set with no more than four steps. These exact decompositions emerge from the fact that mode transformations through second-order unitary operations are linear, and thus, one can utilize matrix representations and matrix decomposition techniques. In fact, for Gaussian operator decompositions, one can find infinitely many elementary sets and decompositions, and instead of those sets above, one may

choose the one that suits best the given situation and purpose.

Universal decompositions.—In order to decompose an arbitrary single-mode operator in CV systems, it has been shown that, in principle, adding a nonlinear element (of order three or more) to the toolbox is sufficient [2]. In the present work we use the following set:

$$\{e^{i(\pi/2)(X^2+P^2)}, e^{it_1X}, e^{it_2X^2}, e^{it_3X^3}\}. \quad (1)$$

This set is not unique, and one may use different Gaussian elements as explained above and a different nonlinear element. However, this particular set turns out to be useful for describing CV quantum computation in the one-way model using CV cluster states [7,8]. Note that one can simplify this elementary set further by omitting the second-order Hamiltonian, since $e^{it^2X^2} = e^{i(2t^4/27)} e^{i(2t/3)P} e^{itX^3} e^{-i(2t/3)P} e^{-itX^3} e^{i(t^3/3)X}$, and using the Fourier transformation whose action is given in Eq. (2). Even though this simplification has value from an academic point of view, as it reduces the minimal number of elementary gates, we are basically motivated by decomposing an arbitrary gate to a set of experimentally accessible gates. All second-order gates are relatively easy to implement and replacing them by third-order Hamiltonians will increase the complexity of the gate sequence. Therefore we shall use the overcomplete set (1) in our decompositions without loss of generality. One may also prefer a further extended set depending on a certain experimental situation in order to reduce the complexity of the decompositions.

In addition, one can obtain *some* nonlinear operations through unitary conjugation: $U e^{iH(a,a^\dagger)} U^\dagger \rightarrow e^{iH(UaU^\dagger, (UaU^\dagger)^\dagger)}$. An important unitary conjugation is the Fourier transform:

$$e^{i(\pi/2)(X^2+P^2)} e^{itX^m} e^{-i(\pi/2)(X^2+P^2)} = e^{itP^m}. \quad (2)$$

Employing unitary conjugation, with the set (1), one can now generate certain nonlinear gates exactly. For example, $e^{itX^3} e^{itP^2} e^{-itX^3} = e^{it(P-t(3/2)X^2)^2}$, which is a fourth-order operator. However, there is only a limited number of such decomposable nonlinear operators, and therefore, for generality, we will make use of the idea of operator approximations.

Besides the abstract notion of universality [2] how can a given unitary exponential operator be decomposed to the elementary set (1) in a systematic and efficient way? Let us first introduce the available toolbox employed to achieve a decomposition as efficient as possible (while efficiency will be defined and discussed later). The tools we use for CV gate decompositions include Gaussian operator decompositions, unitary conjugation, and exponential operator relations as well as approximations. Before proceeding to the general case, let us demonstrate how to realize a particular nonlinear exponential operator using the above tools and the set (1). A very important example is the Kerr interaction operator. It allows you to convert a coherent

state into a cat state [9] and to realize a controlled quantum gate for qubits [10]. The one-mode self-Kerr interaction, up to a Gaussian element, corresponds to $e^{it(X^2+P^2)^2} = e^{it(X^4+X^2P^2+P^2X^2+P^4)}$. In order to decompose the Kerr operation to the set (1), we would first write the full Kerr Hamiltonian as a linear combination of commutators and then realize each through operator approximations. The following relations, together with the Fourier transform (2), are enough to realize this gate up to a phase: $X^4 = -\frac{2}{9}[X^3, [X^3, P^2]]$, $X^2P^2 + P^2X^2 = -\frac{4i}{9}[X^3, P^3]$. Thus, it is sufficient to realize the above commutators and their linear combinations (see supplemental material [4] for more details).

Let us now present the general scheme for an arbitrary Hamiltonian. Obviously, any single-mode Hamiltonian, as a polynomial of bosonic mode operators, consists of operators of the form $cX^mP^n + c^*P^nX^m$. We can show that any such operator can be written as a linear combination of commutation operators. First note that $cX^mP^n + c^*P^nX^m = \text{Re}(c)(X^mP^n + P^nX^m) + i\text{Im}(c)[X^m, P^n]$, and then one can derive the following two identities (see supplemental material [4] for a derivation),

$$X^m = \frac{-2}{3(m-1)}[X^{m-1}, [X^3, P^2]], \quad (3)$$

$$X^mP^n + P^nX^m = -\frac{4i}{(n+1)(m+1)}[X^{m+1}, P^{n+1}] - \frac{1}{n+1} \sum_{k=1}^{n-1} [P^{n-k}, [X^m, P^k]]. \quad (4)$$

Equation (3) is necessary to obtain arbitrary powers of X and P operators with the Fourier conjugation (2), and Eq. (4) basically prescribes how to systematically decompose an elementary Hamiltonian to commutation operations of orders of X and P and their combinations where we can use the tools we have.

For multimode operators one needs an extended elementary set including an entangling operation [2], for example, in the optical context it can be the beam-splitter operation. Using Gaussian decomposition methods, for simplicity, we may assume that we have access to the following gate without loss of generality, $C_Z = e^{2itX_1 \otimes X_2}$. For multimode Hamiltonians we can again use the simplifications for a single mode and Eq. (4), because of the fact that the operators on one mode commute with the operators on the other mode. However, for this purpose, we initially need to realize the two-mode operations with arbitrary powers of X and P in both modes, similar to the single-mode relation (3). The following relation together with Fourier conjugation (2) and Eq. (3), is sufficient to realize the two-mode operations with arbitrary powers of X and P ,

$$P_1^n \otimes P_2^s = -\frac{1}{(n+1)(s+1)} [P_2^{s+1}, [P_1^{n+1}, X_1 \otimes X_2]]. \quad (5)$$

Then, we can use Eq. (4) again with the single-mode operations to realize an arbitrary two-mode expression. As an example, consider the cross-Kerr Hamiltonian (up to a Gaussian transformation): $(X^2 + P^2)_1 \otimes (X^2 + P^2)_2 = X_1^2 \otimes X_2^2 + X_1^2 \otimes P_2^2 + P_1^2 \otimes X_2^2 + P_1^2 \otimes P_2^2$. In this case, the nested commutator $[P_2^3, [P_1^3, X_1 \otimes X_2]]$ will suffice. Another, even simpler example for decomposing a nonlinear two-mode evolution, namely, cubic parametric down-conversion with a quantized pump, is discussed in the supplemental material [4]. In form of a dispersive interaction, it may also be used to mimic a Kerr gate [11].

From an academic perspective, Eqs. (3)–(5) are universal not only for any Hamiltonian, but also for any initial universal set with a nonlinear gate different from X^3 because of the well-known equations: $\frac{\partial F}{\partial P} = -2i[X, F]$, $\frac{\partial F}{\partial X} = 2i[P, F]$, where F is a function of operators X and P . Thus, any initial nonlinear Hamiltonian can be reduced to a form X^m . From a practical perspective, however, it is unwise to use Eqs. (3) and (4) for arbitrary initial sets because of a typical increase of complexity. Instead, one should derive an optimized expression (in terms of the decomposition efficiency, see below) utilizing the available tools for every other Hamiltonian and every other universal set.

Efficiency.—Besides having a systematic method, we also require the decompositions to be relatively efficient. We define efficiency as the number of operators needed to realize, in an approximate fashion, a given operator with a certain negligible error (note that this definition slightly differs from previous ones [2] where efficiency is the scaling of the number of operators with respect to the error). Let us give a few more definitions. If in

$$e^{tC} = e^{t_1A} e^{t_2B} e^{t_3A} \dots e^{t_mB}, \quad (6)$$

the Taylor expansion of both sides matches for the orders of t up to t^m , it is called m th-order decomposition [12]. For example, an important case is when $C = A + B$ for which we will use the term splitting. Another important case is when $C = [A, B]$ which, from now on, we call commutation operator. For example, the identity below is a well-known second-order approximation for a commutation operator. It has been used already in quantum control [13], discrete-variable quantum computation [14], CV quantum computation [2], or, in general, Hamiltonian simulation theory [15],

$$e^{t^2[A,B]} = e^{itB} e^{itA} e^{-itB} e^{-itA} + f(t^3, A, B) + \dots \quad (7)$$

It basically says that, for $t < 1$, the corresponding operator concatenation is the same as applying the commutation operator of A and B , up to some error where the dominant term is of the order t^3 . Now in order to obtain more reliable gates, a straightforward and common way to improve

accuracy is using smaller interaction times, $t \rightarrow t/n$, and applying the decomposition n^2 times to obtain the same interaction time as before,

$$e^{t^2[A,B]} = (e^{iB(t/n)} e^{iA(t/n)} e^{-iB(t/n)} e^{-iA(t/n)})^{n^2} + f\left(\frac{t^3}{n}, A, B\right).$$

Let us call this approach rescaling ([16] and Refs. therein). Besides improving accuracy, rescaling is absolutely necessary to realize nested commutations. For example, one may replace itA by $t^2[B, A]$ in Eq. (7) to simulate the nested commutation operator $e^{it^3[B,[B,A]]} = e^{itB} e^{t^2[B,A]} e^{-itB} e^{-t^2[B,A]} + f(t^4, A, B)$, and similarly for further nested commutations. However, in this identity, an approximation of $[B, A]$ is still needed and eventually, using again Eq. (7), we obtain an operator whose interaction time is of the same order as the dominant error term. In order to obtain a reasonable decomposition, the order of the dominant error should be smaller than t^3 . Thus, again rescaling is needed, requiring relatively many operators to enhance accuracy. For instance, using the approximations described so far, the number of operators to approximate a single commutation operator with coefficient 0.1 and dominant error term $\sim 10^{-3}$ requires 4000 operators, while for the nested commutation operator, with the same values, the number of operators will be $\sim 10^{10}$ (see supplemental material [4]). Hence, better approximations are crucial to achieve reasonable decompositions.

What we propose to use is a general and powerful method for obtaining higher-order approximations in a direct fashion (details on this rather technical part of our proposal can be found in the supplemental material [4]; see also, for instance, page 393 of Ref. [17]). In this approach, concatenations converge much faster to an arbitrary set of commutations and linear combinations of these, reducing the number of operators from the order of 10^{10} to the order of tens.

Experiment.—An immediate consequence of our work is that it bridges the originally huge gap between abstract notions of CV decomposition theory and possible experimental implementations, for instance, within quantum optics. Highly nonlinear quantum gates (such as quartic Kerr gates) may then be realized in a deterministic fashion by concatenating tens of quadratic and cubic gates. Figure 1 illustrates two experimental schemes. They both combine ideas for conditional optical state preparation, in order to probabilistically prepare and distill high-fidelity cubic phase states $\approx \int dx e^{itx^3} |x\rangle$ [18–20], with those for temporally encoded [21,22] and fully homodyne-detection-based [23] CV cluster computation. The complication of realizing nonlinear gates is shifted offline into the preparation of the cubic ancillae.

In Fig. 1(a), the conditional state preparer (CSP) is linked with a quantum memory (QM) through a classical channel in order to signal whenever an offline state is available such that an optical pulse carrying the latest

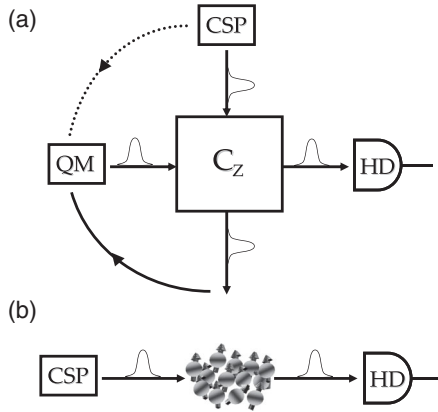


FIG. 1. Concatenating elementary gates using offline conditional state preparation (CSP) of squeezed and cubic states, a quantum memory (QM), an optical C_Z gate, $e^{2i\mu X_1 \otimes X_2}$, and homodyne detection (HD); solid lines represent the optical paths. In (a), an optical quantum state is released from the QM only when the CSP succeeds (dotted line is classical feedforward). In (b), quantum information storage and processing go hand in hand by employing suitable quadratic light-matter interactions between optical pulses and an atomic ensemble.

quantum information is released from the quantum memory. After the first application of the optical C_Z gate between a first input pulse coming from the left and a first ancilla pulse coming from the top (emerging from the CSP), the C_Z -transformed input pulse is measured through homodyne detection (HD), while the C_Z -transformed ancilla pulse is sent to the quantum memory with its quantum state transferred onto the memory. After preparation of the next ancilla pulse, a new input pulse emerges from the memory for a second application of the C_Z gate and so on. This is similar to the scheme of Ref. [21] for a single quantum wire (corresponding to a linear CV cluster chain), but this time including cubic states and quantum memories, and excluding nonlinear measurements such as photon counting (except for the CSP). Note that storage of nonclassical states has been already demonstrated experimentally [24,25].

The scheme in Fig. 1(b) uses an atomic ensemble and a quadratic light-matter interaction [26]. The light-matter interaction is always delayed until an appropriate offline state is available. Quantum information can be stored and processed at the same time within the ensemble [22] by swapping the optical and atomic states after every interaction, simply using additional quadratic interactions [27,28]. As opposed to Refs. [21,22], the optical state is either a squeezed vacuum or a conditionally prepared cubic state, inserted into the temporal cluster chain whenever needed [23]. Thus, we make explicit use of the atomic memory to compensate for the probabilistic optical ancilla-state preparations. Only with these cubic ancillae is it possible to perform universal gates solely through homodyne detection [23]. In principle, our decomposition method would then determine what homodyne

local-oscillator angles to choose in order to realize a given Hamiltonian in a deterministic fashion. Currently available squeezing levels of almost 13 dB are, in principle, sufficient to perform up to 73 elementary teleportations in a nonclassical fashion; four such elementary Gaussian gates have been implemented already in a fully optical, spatial encoding using 5.5 dB-squeezed light sources [29].

In summary, we proposed a systematic framework to decompose arbitrary CV unitaries (from arbitrary multi-mode Hamiltonians) into an experimentally realizable elementary gate set. Different from previous proof-of-principle demonstrations, our treatment brings the abstract notions of decomposition theory for CV quantum computation close to experimental implementations.

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