Effects of Interaction on Quantum Spin Hall Insulators

Dung-Hai Lee
Department of Physics, University of California at Berkeley, Berkeley, California 94720, USA
Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA
(Received 15 June 2011; published 14 October 2011)

We study the S_z-conserving quantum spin Hall insulator in the presence of Hubbard U from a field theory point of view. The main findings are the following. (1) For arbitrarily small U the edges possess power-law correlated antiferromagnetic XY local moments. Gapless charge excitations arise from the Goldstone-Wilczek mechanism. (2) Electron tunneling between opposite edges allows vortex instantons to proliferate when K, the XY stiffness constant, satisfies $4\pi K+(4\pi K)^{-1}<4$. When the preceding inequality is violated, the edge modes remain gapless despite the sample width being finite. (3) The phase transition from the topological insulator to the large U antiferromagnetic insulator is triggered by the condensation of magnetic excitons. (4) In the large U antiferromagnetic insulating phase the magnetic vortices carry charges proportional to the square magnitude of the antiferromagnetic order parameter.

DOI: 10.1103/PhysRevLett.107.166806 PACS numbers: 73.43.−f, 03.65.Vf, 71.10.Pm, 72.25.Hg

The subject of topological insulators (TI) has attracted considerable attention recently [1]. A signature of this type of band insulator is the presence of itinerant boundary states in the bulk band gap [2–6]. Moreover, unlike those in usual band insulators, these itinerant in-gap states are robust against any modification of the free-electron Hamiltonian so long as they (i) respect time-reversal symmetry, and (ii) do not close the bulk band gap. Because of the robustness, these boundary states can evade Anderson's localization in the presence of (time-reversal invariant) disorder [7]. At the present time noninteracting TIs are fairly well understood. What remains open is the effect of electron-electron interaction on TIs [8].

Recently, two independent Monte Carlo simulations [9,10] were performed on the simplest kind of two dimensional interacting TI. The Hamiltonian studied in these works is H_0+H_u where H_0 is the S_z-conserving free-electron model introduced by Kane and Mele [2]:

$$H_0=\sum_{\sigma=\pm1}\Big\{-\sum_{\langle ij\rangle}c^+_{i\sigma}c_{j\sigma}+it'\sum_{\langle\langle ij\rangle\rangle}\sigma\nu_{ij}c^+_{i\sigma}c_{j\sigma}\Big\}. \quad (1)$$

Here i, j label the sites of a honeycomb lattice, the first term describes the nearest neighbor hopping, and the second term is a spin-dependent second neighbor hopping. Here $\nu_{ij}=(\hat d_1\times\hat d_2)_z/|(\hat d_1\times\hat d_2)_z|$ where $\hat d_1$ and $\hat d_2$ are unit vectors along the two bonds the electron traverses when hopping from j to i. H_u is given by $H_u=U\sum_i n_{i\uparrow}n_{\downarrow}$. According to Ref. [9], for $t'\gtrsim 0.03$ (see Fig. 1) there are only two phases as a function of U. The large U phase is an easy-plane (XY) antiferromagnetic (AF) Mott insulator; at small U it is an (interacting) TI with gapless spin and charge edge excitations.

The present work is motivated by the following considerations. Consider a system with edges [Fig. 2(a)]. At $U=0$ the bulk has a band gap and the only low energy excitations are the "helical" edge modes described by the following Hamiltonian

$$H_{E0}=\mp iv\int dx\Psi^+\sigma_z\partial_x\Psi, \quad (2)$$

where $\mp$ applies to the top or bottom edges. In Eq. (2) v is the edge velocity (which will be set to 1 in the rest of the Letter), and Ψ is a two component fermion field whose first (second) component corresponds to spin up (down). For small U the bulk is nonmagnetic [9,10] but the edge will develop local moments. To see that we Hubbard-Stratonavich decouple the Hubbard U term in the imaginary-time Euclidean action as

$$e^{-U\int d\tau\sum_j n_{j\uparrow}n_{j\downarrow}}\sim\int D[S^\pm]e^{-U\int d\tau\sum_j[S^+_jS^-_j-S^+_jc^+_{j\downarrow}c_{j\uparrow}-S^-_jc^+_{j\uparrow}c_{j\downarrow}]}. \quad (3)$$

We note that upon time reversal $S^\pm_j\to-S^\mp_j$ and $c^+_{j\sigma}c_{j-\sigma}\to -c^+_{j-\sigma}c_{j\sigma}$, and hence, the decoupled action is time-reversal invariant. (Note that the imaginary-time τ transforms as it, hence, is invariant under time reversal.) In mean-field

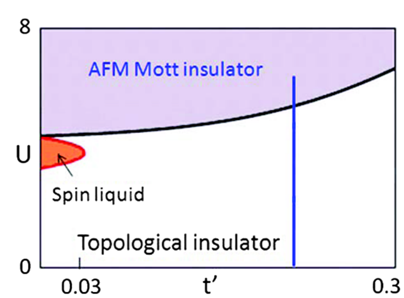

FIG. 1 (color online). A schematic reproduction of the phase diagram of the H_0+H_u reported in Ref. [9]. The vertical cut is considered in the text.

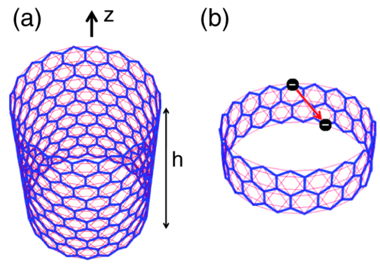

FIG. 2 (color online). (a) The spin Hall insulator defined on a cylinder with height h. The thick and thin bonds denote the nearest and second neighbor hopping. (b) When the cylinder is short, the electron can directly tunnel from one edge to the other.

theory $S^{\pm}$ takes on space-time independent values (hence, spontaneously break the time-reversal symmetry) which introduce two mass terms into H_0:

$$H_{E0} - Um\left[\cos\theta_0 \int dx \Psi^{\dagger}\sigma_x\Psi + \sin\theta_0 \int dx \Psi^{\dagger}\sigma_y\Psi\right]. \tag{4}$$

Because the noninteracting theory has a logarithmically diverging susceptibility with respect to these mass terms mean-field theory predicts the edges will have long-range AF XY order for arbitrarily small positive U. Of course, such long-range order is destroyed by spin wave fluctuations and the time-reversal symmetry is restored. The resulting edges have power-law decaying AF XY correlation. However, since the local moments introduce a single particle gap, one might wonder where are the gapless charge excitations.

First, to make sure our statement concerning AF moment formation at infinitesimal U is correct for the edges of a two dimensional TI, we perform a mean-field calculation on a cylinder for $t' = 0.2$ and $0 \le U \le 5$ (this corresponds to the cut associated with the blue line interval in Fig. 1). The results are shown in Fig. 3; from which it is clear that while the order parameter deep in the bulk (the blue curve) vanishes for $U \lesssim 2.4$, the edge order parameter (the red curve) survives to the lowest U value. There are two types of fluctuations above the mean-field vacuum: the massive modulus fluctuation in m and the massless Goldstone mode (i.e., spin wave) fluctuations. In the presence of these fluctuations we need to replace Eq. (4) by the following action $S_E = \int dx d\tau \mathcal{L}_E$:

$$\begin{aligned}\mathcal{L}_E = {}& \bar{\Psi}\partial_\tau\Psi \mp i\Psi^{+}\sigma_z\partial_x\Psi - U\bar{m}[\cos\theta(x,\tau)\Psi^{\dagger}\sigma_x\Psi \\ & + \sin\theta(x,\tau)\Psi^{\dagger}\sigma_y\Psi] + U[\delta m(x,\tau)^2 \\ & - \delta m(x,\tau)\Psi^{\dagger}\sigma_x\Psi].\end{aligned} \tag{5}$$

In the following we first ignore the δm fluctuation and focus on the effect of the spin waves. In Ref. [11] Goldstone and Wilczek showed that integrating out the gapped fermions in Eq. (5) in the presence of a background electromagnetic gauge field and the total effective action looks like

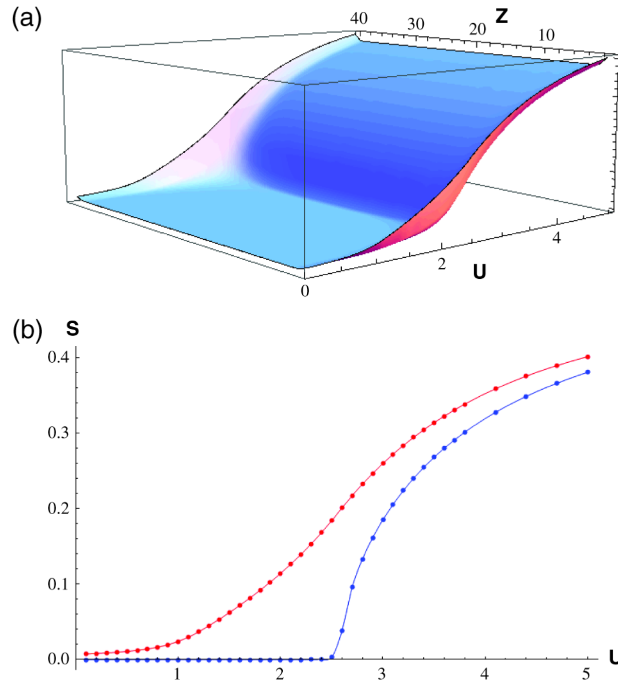

FIG. 3 (color online). (a) The mean-field antiferromagnetic XY order parameter as a function of z [see Fig. 2(a) and U]. The calculation is done for a cylinder with 35 unit cells in the periodic direction and 40 unit cells in the z direction. The value t' is 0.2. (b) The order parameter as a function of U deep in the bulk $z = 20$ (lower curve), and at the edge $z = 1$ or 40 (upper curve). It is clear that while the order parameter in the bulk vanishes for $U \lesssim 3$ (the small rounding is due to finite size effect), the edge order parameter survives to the lowest U value.

$$S_{\rm eff} = \int dx d\tau \left\{\frac{K}{2}(\partial_\mu\theta)^2 - \frac{ie}{2\pi}(A_0\partial_x\theta - A_x\partial_\tau\theta)\right\}. \tag{6}$$

Here e is the electron charge, and the last two terms are the famous chiral anomaly. Using the method of Abanov and Wiegmann [12] we have derived the stiffness term $K = (m^2/2\pi)\int_0^\Lambda \frac{pdp}{(p^2+m^2)^2} = (1 - \frac{m^2}{\Lambda^2+m^2})/4\pi$ (here Λ is a momentum cutoff). Hence as $\Lambda/m \to \infty$ $K \to 1/4\pi$.

So far in deriving K we have ignored the δm fluctuation. The easiest way to account for these fluctuations is to replace the fermion bilinears in Eq. (5) by their bosonization formulas [13]:

$$\begin{aligned}\mathcal{L}_E = {}& \tfrac{1}{2}(\partial_\mu\phi)^2 - U\bar{m}C_\Lambda\cos[\theta(x,\tau) - \sqrt{4\pi}\phi(x,\tau)] \\ & + U[\delta m(x,\tau)^2 - \delta m(x,\tau)C_\Lambda\cos(\sqrt{4\pi}\phi)],\end{aligned} \tag{7}$$

where C_Λ is a cutoff dependent constant. Now it is straightforward to integrate out δm and ϕ. The results are (i) the generation of the term $\sim \int dx d\tau \cos 2\theta$ which is irrelevant for $(\pi K)^{-1} < 2$, (ii) an order U correction to K: $K \to K + O(U)$. Hence for large Λ and small U the effect of δm fluctuations is to make Eq. (6) the action of a repulsive Luttinger liquid [14].

In the absence of A_μ, Eq. (6) implies the XY correlation function

$$\langle e^{i\theta(0,0)} e^{-i\theta(x,\tau)}\rangle \sim (x^2+\tau^2)^{-1/4\pi K}; \qquad (8)$$

hence, as claimed earlier, the long-range order in mean-field theory is destroyed. The last two terms of Eq. (6) imply the space and time gradients in θ produce excess charge and current densities at the edges:

$$\rho_E = \frac{e}{2\pi}\partial_x\theta, \qquad J_E = -\frac{e}{2\pi}\partial_t\theta. \qquad (9)$$

Because of Eq. (9) gapless spin wave excitations induce charge and current density fluctuations with the following power-law correlation functions:

$$\Pi_{\rho/J}(x,t) \propto \langle \partial_{x/t}\theta(0,0)\partial_{x/t}\theta(x,t)\rangle \sim \pm\frac{t^2-x^2}{(x^2+t^2)^2}.$$

Thus through the Goldstone-Wilczek mechanism gapless charge excitations emerge.

The next question concerns the space-time vortices (instantons) of the θ field. Because of Eq. (9) a vorticity-m instanton at the space-time location $(x_0,\ t_0)$ will cause $\oint_{\partial D} dx_\mu \partial_\mu\theta = -(2\pi/)\oint_{\partial D} dx_\mu \epsilon^{\mu\nu}J_{E,\nu} = -(2\pi/e)\int_D d^2x\partial_\mu J_{E,\mu} = 2\pi m$. Here D is an arbitrary disk containing $(x_0,\ t_0)$ and $J_{E,\mu} = (\rho_E, J_E)$ is the edge 2-current. This implies

$$\partial_t\rho_E + \partial_x J_E = -me\delta(x-x_0)\delta(t-t_0); \qquad (10)$$

hence vortex instantons violate the *edge* charge conservation and, therefore, under usual circumstances should be forbidden. Nonetheless, such instantons can occur through the tunneling of electrons from one edge to the other [15] [Fig. 2(b)]. Annihilating a right (left) moving electron at the edge removes a charge e and spin $1/2(-1/2)$. Hence

$$\psi_{R/L}(x) \sim \exp\left\{\mp i\theta(x)/2 + i2\pi\int_x^\infty dy\Pi(y)\right\}, \qquad (11)$$

where $[\theta(x), \Pi(y)] = i\delta(x-y)$. The above result resembles the usual 1D bosonization formula. This is not surprising since if we identify θ with $2\sqrt{\pi}\phi$, Eq. (9) becomes the bosonization expression for the charge and current densities. It can be shown straightforwardly that for $4\pi K + (4\pi K)^{-1} < 4$ the interedge electron tunneling is a relevant perturbation. Under such a condition, interedge electron tunneling will gap the edge modes. However, if $4\pi K + (4\pi K)^{-1} > 4$ the interedge electron tunneling is irrelevant. In that case the quantum spin Hall effect will survive even when h, the sample width, is finite. Such stabilization of the TI state for finite width sample is a pure interaction effect.

In the interacting TI phase the system exhibits quantized spin Hall conductance. This can be understood as follows. In the presence of an electric field between the two edges, a voltage difference V develops. This induces a difference in $\partial_x\theta$ between the two edges (E_1 and E_2) $(\partial_x\theta)_{E_1} - (\partial_x\theta)_{E_2} = \frac{eV}{2\pi K}$. Because the spin current is $K\partial_x\theta$, this gives $J_{S_z}^{\rm tot} = \frac{e}{2\pi}V$; hence the spin Hall conductance is $\frac{e}{2\pi}$ which is the same as the free-electron value. As to the two terminal conductance, it was pointed out in the context of quantum wires that in the presence of translation invariance interaction effect does not change the two terminal conductance of a Luttinger liquid [16].

Finally, we consider the bulk transition between the AF Mott insulator and the TI. For this discussion let us use the periodic boundary condition, and consider the blue cut in Fig. 1. First we approach the Mott insulator from the TI side. In the presence of U its lowest-energy exciton is magnetic [we do not use the word "triplet" because $SU(2)$ is broken by the spin-orbit hopping down to $U(1)$]. An example of the XY order parameter profile associated with a magnetic exciton is shown in Fig. 4. Upon increasing U the transition into the AF insulator is triggered by the condensation of magnetic excitons. At the transition the modulus and the phase coherence of the AF XY order parameter develop simultaneously.

It is also instructive to approach the transition from the AF Mott insulator side. In this case one naturally expects the XY order to be destroyed by the condensation of vortices [17]. Because the AF XY order parameter, the triplet superconducting order parameter, and the quantum spin Hall order parameter [which introduces the spin-dependent hopping term in Eq. (1)] form a Wess-Zumino-Witten five-tuplet for the free-graphene band structure [18], one expects the following charge density-Skyrmion density relation [12]

$$\rho = \frac{1}{2\pi}\epsilon_{abc}n^a\partial_x n^b\partial_y n^c. \qquad (12)$$

Here $n^{1,2,3}$ are the components of the unit vector, with n^1 associated with the quantum spin Hall order parameter [the strength of the imaginary second neighbor hopping in Eq. (1), which is fixed] and $n^{2,3}$ associated with the AF XY order parameters. The vortex charge is therefore proportional to $n_1(1-n_1^2)$. Since the XY order parameter

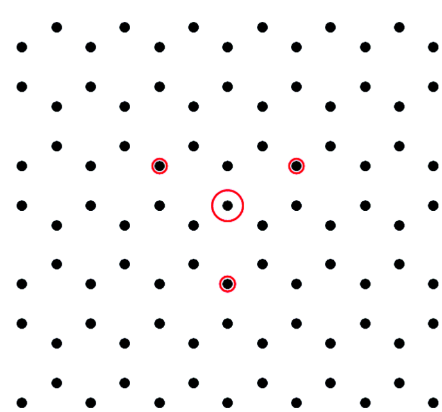

FIG. 4 (color online). An example of the XY order parameter profile associated with a localized magnetic exciton. The size of the open circles is proportional to the magnitude of the order parameter.

vanishes at the Mott → TI transition, the condensed vortices are charge neutral. Were it not true the vortex condensed phase cannot be an insulator. In addition, since the modulus of the XY order parameter vanishes at the transition the vortices do not see a background magnetic flux which frustrates the vortex condensation. This implies the universality class of the transition is three-dimensional XY, as claimed in Ref. [9].

In summary, despite the apparent differences, the power-law correlated antiferromagnetic XY edges do exhibit properties expected for the quantum spin Hall insulators. The essential physics is the Goldstone-Wilczek mechanism; through which the space-time gradients of the phase angle of the XY order parameter are proportional to the charge and current densities. The space-time vortices of the XY order parameter violate edge charge conservation and, hence, are prohibited in thermodynamic samples. This is the mechanism through which the gapless charge and spin excitation are protected at the edges. At the moment we do not have a good picture for the ''spin liquid dome'' in Fig. 1. The main findings of this Letter are summarized in the abstract.

I thank Tao Xiang, Guang-Ming Zhang, Shinsei Ryu, and Hong Yao for helpful discussions. I am particularly grateful to Cenke Xu for explaining how to view the Mott to TI transition from the Mott insulating side to me. I acknowledge the support by the DOE Grant No. DE-AC02-05CH11231.

[1] J. Moore, Nature Phys. **5**, 378 (2009); M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010); X.-L. Qi and S.-C. Zhang, Phys. Today **63**, No. 1, 33 (2010).
[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005); **95**, 146802 (2005).
[3] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[4] M. Konig, S. Wiedmann, C. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).
[5] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[6] D. Hsieh *et al.*, Nature (London) **452**, 970 (2008); D. Hsieh *et al.*, Science **323**, 919 (2009); Y. Xia *et al.*, Nature Phys. **5**, 398 (2009); Y. L. Chen *et al.*, Science **325**, 178 (2009).
[7] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[8] D. Pesin and L. Balents, Nature Phys. **6**, 376 (2010); C. N. Varney, K. Sun, M. Rigol, and V. Galitski, Phys. Rev. B **82**, 115125 (2010); S. Rachel and K. Le Hur, Phys. Rev. B **82**, 075106 (2010); D. Soriano and J. Fernández-Rossier, Phys. Rev. B **82**, 161302 (2010); J. Goryo and N. Maeda, J. Phys. Soc. Jpn. **80**, 044707 (2011).
[9] M. Hohenadler, T. C. Lang, and F. F. Assaad, Phys. Rev. Lett. **106**, 100403 (2011).
[10] D. Zheng, C. Wu, and G.-M. Zhang, arXiv:1011.5858.
[11] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[12] A. G. Abanov and P. B. Wiegmann, Nucl. Phys. **B570**, 685 (2000).
[13] R. Shankar, Acta Phys. Pol. B **26**, 1835 (1995).
[14] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).
[15] One might argue that an alternative tunneling process where an edge electron tunnel into the bulk and back can also allow vortex instantons. However, in such virtual processes the intermediate energy cost due to the bulk charge gap confines the instanton and antiinstanton.
[16] Y. Oreg and A. M. Finkelstein, Phys. Rev. B **54**, R14265 (1996).
[17] M. P. A. Fisher and D.-H. Lee, Phys. Rev. B **39**, 2756 (1989).
[18] S. Ryu, C. Mudry, C.-Y. Hou, and C. Chamon, Phys. Rev. B **80**, 205319 (2009).